Solution 4

1. (a) Hessian of f is

$$H = \begin{bmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$$

Note that $H_{11}, H_{22} \geq 0$, $\det(H) = 4x^2/y^4 - 4x^2/y^4 = 0$. By Sylvester's criterion, H is positive semidefinite on the $\mathbb{R} \times (0, \infty)$, so f is convex.

(b) Hessian of f is

$$H = \frac{e^{x+y}}{(e^x + e^y)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which is positive semidefinite, so f is convex.

2. (a) Hessian of f being positive definite implies f being strictly convex, but the converse is not true. For example, $f(x) = |x|^3$ is convex but $f''(0) = 0$.

(b)-(d) are true. We only give the proof of (b). The proof of (c) and (d) are similar.

Suppose $f(y) > f(x) + \nabla f(x)(y - x)$ for every $x, y \in \Omega$. Let $x, y \in \Omega andt \in [0, 1]$. Let $z = tx + (1 - t)y$. We have

$$f(x) > f(z) + \nabla f(z)(x - z)$$
$$f(y) > f(z) + \nabla f(z)(y - z)$$

Then

$tf(x) + (1 - t)f(y) > f(z) + \nabla f(z)(tx + (1 - t)y - z) = f(z)$.

For the converse, we have $f(x + (y - x)t) < f(x) + t(f(y) - f(x))$ for all $t \in [0, 1]$, so $f(y) - f(x) > \frac{f(x+(y-x)t)-f(x)}{t}$ for all $t \in (0, 1]$. Taking $t \to 0$, we have $f(y) - f(x) > \nabla f(x)(y - x)$.

3. (a)

$$f(x) = \begin{cases} x^2 - 2x - 3 & x \in (-\infty, -1) \\ 0 & x \in [-1, 1] \\ x^2 + 2x - 3 & x \in (1, \infty) \end{cases}$$

Subdifferential at $x \in (-\infty, -1)$ is $\{2x - 2\}$, at $x \in (1, \infty)$ is $\{2x + 2\}$, at $x \in (-1, 1)$ is $\{0\}$, at $x = -1$ is $[-4, 0]$, at $x = 1$ is $[0, 4]$.

(b) Sorry that there is a mistake, we should consider $f(x) = \sqrt{|x|}$. Note that

$$y \in (\partial f)(0) \Longleftrightarrow f(z) \geq f(0) + y \cdot (z - 0) \quad \forall z \in \mathbb{R}.$$

For $f(x) = \sqrt{|x|}$ this thus becomes

$$y \in (\partial f)(0) \Longleftrightarrow \sqrt{|z|} \geq y \cdot z \quad \forall z \in \mathbb{R}.$$

Since there is no such y , which means $(\partial f)(0) = \{0\}$.