Solution 3

1.

(a) For any $t \in [0,1]$,$x, y \in \mathbb{R}^N$

$$f(A(tx + (1-t)y)) = f(tAx + (1-t)Ay) \le tf(Ax) + (1-t)f(Ay)$$

Thus, g is convex.

(b) The claim follows immediately from the fact that that $\text{epi}(\sup_\lambda f_\lambda) = \cap_\lambda \text{epi}(f_\lambda)$.

2.

(a) A convex but not strictly convex function is $f(x) = x$. A strictly convex but not strongly convex function is $f(x) = x^{3/2}, x \in \mathbb{R}^+$.

(b) Counter example: $f(x) = -x$ and $g(x) = x^2$ are both convex but $f \circ g = -x^2$ is not.

3.

(a) First consider the case $\alpha = 1$, which easily gives convexity. Then consider the case $\alpha > 1$, where $f(x) = |x|^\alpha$ is differentiable. Then $f''(x) = \alpha(\alpha - 1)x^{\alpha-2} \ge 0$ for $x \ge 0$ and $f''(x) = \alpha(\alpha-1)(-x)^{\alpha-1} \ge 0$ for $x \le 0$. Thus, f is convex.

(b) Note that the Hessian matrix is $\text{diag}\{1/\left(1+x_1\right)^2, 1/\left(1+x_2\right)^2, \ldots, 1/\left(1+x_N\right)^2\}$, which is positive definite.

(c) Let P be semi-positive definite. Let $t \in [0,1]$ and $x, y \in \mathbb{R}^N$. We have

$$\begin{aligned}
&(tx + (1-t)y)^\top P(tx + (1-t)y) - tx^\top Px - (1-t)y^\top Py \\
=& -t(1-t)x^\top Px - t(1-t)y^\top Py + t(1-t)x^\top Py + t(1-t)y^\top Px \\
=& -t(1-t)\left(x^\top Px + y^\top Py - x^\top Py - y^\top Px\right) \\
=& -t(1-t)(x-y)^\top P(x-y) \\
<& 0.
\end{aligned}$$

Thus, $x^\top Px$ is convex.