Homework 2

1. (a) $\text{int}(A) = \varnothing, \text{ri}(A) = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$;
$\text{int}(B) = \text{ri}(B) = B$

(b) It is true that $T \subseteq S$ implies $\text{int}(T) \subseteq \text{int}(S)$. For any $x \in \text{int}(T)$, there exists an open set $U \subseteq T$ containing x. Then $x \in U \subseteq T \subseteq S$, which implies $x \in \text{int}(S)$.

It is NOT true that $T \subseteq S$ implies $\text{ri}(T) \subseteq \text{ri}(S)$. For example, let $T = \{(x, 0) \in \mathbb{R}^2, 0 \leq x \leq 1\}$ and $S = \{(x, y) \in \mathbb{R}^2, 0 \leq x, y \leq 1)\}$ in $\mathbb{R}$. We have $\text{ri}(T) = \{(x, 0) \in \mathbb{R}^2, 0 < x < 1\}$ and $\text{ri}(S) = S = \{(x, y) \in \mathbb{R}^2, 0 < x, y < 1)\}$.

2. (a) $f_1 + f_2$ is convex. For any $x, y \in \mathbb{R}^N$, $t \in [0, 1]$. We have

$$\begin{aligned}
&t\,(f_1 + f_2)\,(x) + (1 - t)\,(f_1 + f_2)\,(y) \\
=&t f_1(x) + (1 - t) f_1(y) + t f_2(x) + (1 - t) f_2(y) \\
\geq& f_1(tx + (1 - t)y) + f_2(tx + (1 - t)y) \\
=&\,(f_1 + f_2)\,(tx + (1 - t)y).
\end{aligned}$$

(b) $f_1 \cdot f_2$ may not be convex. For example, $f_2(x) = -1$, $f_1(x) = x^2$.

(c) $f_1 - f_2$ may not be convex. The counter example in (b) is also a counter example for this.

(d) $\max(f_1, f_2)$ is convex. For any $x, y \in \mathbb{R}^N, t \in [0, 1]$. For $i = 1, 2$, we have

$$\begin{aligned}
f_i(tx + (1 - t)y) &\leq t f_i(x) + (1 - t) f_i(y) \\
&\leq t \max(f_1, f_2)\,(x) + (1 - t) \max(f_1, f_2)\,(y).
\end{aligned}$$

3. Suppose f is convex. For any $(x_1, t_1), (x_2, t_2) \in \text{epi}(f)$ and $\theta \in [0, 1]$, we have

$$y := \theta\,(x_1, t_1) + (1 - \theta)\,(x_2, t_2) = (\theta x_1 + (1 - \theta) x_2, \theta t_1 + (1 - \theta) t_2)$$

Since f is convex, we have

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2) \\ \leq \theta t_1 + (1-\theta) t_2,$$

which implies $p \in \text{epi}(f)$ and $\text{epi}(f)$ is convex.

On the other hand, suppose $\text{epi}(f)$ is convex. Let $x_1, x_2 \in \mathbb{R}^N, \theta \in [0,1]$. Note that $(x_1, f(x_1)), (x_2, f(x_2)) \in \text{epi}(f)$. Since $\text{epi}(f)$ is convex, we have

$$(\theta x_1 + (1-\theta)x_2, \theta f(x_1) + (1-\theta) f(x_2)) \\ = \theta (x_1, f(x_1)) + (1-\theta)(x_2, f(x_2)) \in \text{epi}(f)$$

Then $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$, which implies that f is convex.

For question 2(d), take any $(x,t) \in \mathbb{R}^{N+1}$,

$$(x,t) \in \text{epi}(f_1) \cap \text{epi}(f_2) \\ \Longleftrightarrow f_1(x) \leq t \text{ and } f_2(x) \leq t \\ \Longleftrightarrow \max(f_1, f_2)(x) \leq t \\ \Longleftrightarrow (x,t) \in \text{epi}(\max(f_1, f_2))$$

Thus, $\text{epi}(\max(f_1, f_2)) = \text{epi}(f_1) \cap \text{epi}(f_2)$. Since $\text{epi}(f_i), i = 1,2$ is convex, $\text{epi}(\max(f_1, f_2))$ is also convex. Thus, $\max(f_1, f_2)$ is convex.