Homework 1

1. Denote the three sets in (a) (b) and (c) as A,B and C respectively. Note that a set is convex (affine) if and only if its convex (affine) hull is itself.

(a) For any $x, y \in A$, let $x = x_0 + \theta_x v$ and $y = x_0 + \theta_y v$. Then,

$$(1-\alpha)x + \alpha y = (1-\alpha)\left(x_0 + \theta_x v\right) + \alpha\left(x_0 + \theta_y v\right) = x_0 + \left(1 - \alpha\theta_x + \alpha\theta_y\right) v.$$

Since $1 - \alpha\theta_x + \alpha\theta_y \geq 0$ if $0 \leq \alpha \leq 1$, A is convex.

The affine hull of A is

$$\left\{x \in \mathbb{R}^N : x = x_0 + \theta v, \theta \in \mathbb{R}\right\}.$$

Hence, A is not affine.

(b) $\forall x, y \in B$, since $\|x\|, \|y\| \leq 1$, we have

$$\|(1-\alpha)x + \alpha y\| \leq |1-\alpha||x|| + |\alpha|||y|| \leq |1-\alpha| + |\alpha| \leq 1.$$

Hence, B is convex.

The affine hull of B is

$$\{(x_1, x_2, \ldots, x_{N-1}, 0) : x_1, \ldots, x_{N-1} \in \mathbb{R}\}.$$

Hence, B is not affine.

(c) Pick any $x, y \in C$, we have

$$A((1-\alpha)x + \alpha y) = (1-\alpha)Ax + \alpha Ay = (1-\alpha)b + \alpha b = b.$$

Hence, C is convex and affine.

2. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be convex sets and set $C = \bigcap_{\lambda\in\Lambda} C_\lambda$. For any $x, y \in C$. We have $x, y \in C_\lambda$ for any $\lambda \in \Lambda$. Since each C_λ is convex , we have $(1-\alpha)x + \alpha y \in C_\lambda$ for any $\lambda \in \Lambda$. Hence, $(1-\alpha)x + \alpha y \in C = \bigcap_{\lambda\in\Lambda} C_\lambda$.

3. ($\Rightarrow$): Suppose the cone C is convex. Pick any $x = x_1 + x_2 \in C + C$ for some $x_1, x_2 \in C$. We have $\frac{x_1+x_2}{2} \in C$. Since C is a cone, we have

$$x = x_1 + x_2 = 2 \cdot \frac{x_1 + x_2}{2} \in C.$$

($\Leftarrow$) : Suppose $C + C \subset C$. Pick any $x, y \in C$. Let $\alpha \in [0, 1]$. Since C is a cone, we have $\alpha x, (1-\alpha)y \in C$. Then $\alpha x + (1-\alpha)y \in C$ and C is convex.