1 Conjugate Function

1.1 Extended Real-valued functions

Sometimes, we may allow functions to take infinite values. For example the indicator function of a set X defined by

$$\delta_X(x) = \begin{cases} 0 & x \in X \\ \infty & x \notin X \end{cases}$$

These functions are characterize by their epigraph.

The *epigraph* of a function $f : X \to [-\infty, \infty]$, where $X \subset \mathbb{R}^n$, is given by

$$\text{epi}(f) = \{(x, w) | x \in X, w \in \mathbb{R}, f(x) \leqslant w\}.$$

The *effective domain* of f is given by

$$\text{dom}(f) = \{x | f(x) < \infty\}.$$

Note that $\text{dom}(f)$ is just the projection of $\text{epi}(f)$ on $\mathbb{R}^n$.

We also say f is *proper* if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$. We say f is *improper* if it is not proper. By considering $\text{epi}(f)$, it means that $\text{epi}(f)$ is not empty and does not conatin any vertical line.

Next, we will extend our definition of convexity to extended real-valued function. Since the sum $-\infty + \infty$ is not well defined, we cannot follow the definition in the real-valued case. The epigraph provides us a way to deal with this.

We say a extended real-valued function $f : \mathbb{R}^n \to [-\infty, \infty]$ is *convex* if $\text{epi}(f) \subset \mathbb{R}^{n+1}$ is convex.

If the epigragh of a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is closed, we say that f is a *closed* function.

1.2 Conjugate Function

Consider a function $f : \mathbb{R}^n \to [-\infty, \infty]$. The *conjugate function* of f is the function $f^* : \mathbb{R}^n \to [\infty, \infty]$ defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}$$

Remark: f^* is convex even if f is not convex.

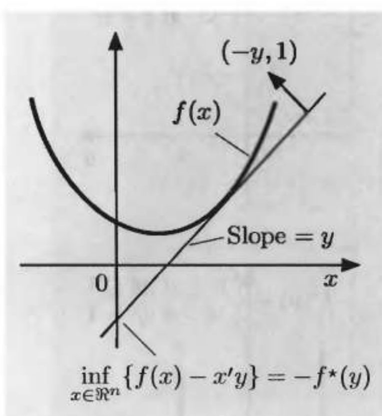

Figure 1.6.1. Visualization of the conjugate function

$$f^\star(y) = \sup_{x \in \Re^n} \left\{x'y - f(x)\right\}$$

of a function f. The crossing point of the vertical axis with the hyperplane that has normal $(-y, 1)$ and passes through a point $\left(\overline{x}, f(\overline{x})\right)$ on the graph of f is

$$f(\overline{x}) - \overline{x}'y.$$

Thus, the crossing point corresponding to the hyperplane that supports the epigraph of f is

$$\inf_{x \in \Re^n} \{f(x) - x'y\},$$

which by definition is equal to $-f^\star(y)$.

Examples of conjugate functions

1. $f(x) = ||x||_1$

$$\begin{aligned} f^*(a) &= \sup_{x \in \mathbb{R}^n} \langle x, a \rangle - ||x||_1 \\ &= \sup \sum (a_n x_n - |x_n|) \\ &= \begin{cases} 0 & ||a||_\infty \leq 1 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

2. $f(x) = ||x||_\infty$

$$\begin{aligned} f^*(a) &= \sup_{x \in \mathbb{R}^n} \sum a_n x_n - \max_n |x_n| \\ &\leq \sup \sum |a_n||x_n| - \max_n |x_n| \\ &\leq \max_n |x_n| ||a||_1 - \max_n |x_n| \\ &\leq \sup ||x||_\infty (||a||_1 - 1) \\ &= \begin{cases} 0 & ||a||_1 \leq 1 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

If $||a||_1 \leq 1$, $\langle 0, a \rangle - ||0||_\infty = 0$, $f^*(a) \geq 0$ in this case.
If $||a||_1 > 1$, then $\langle x, a \rangle - ||x||_\infty$ is unbounded. Hence

$$f^*(a) = \begin{cases} 0 & ||a||_1 < 1 \\ \infty & \text{otherwise} \end{cases}$$

We can also consider the conjugate of f^* (double conjugate of f). It is given by

$$f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{\langle y, x \rangle - f^*(y)\}$$

It is natural to ask whether $f = f^{**}$. Indeed, this is true under some conditions.

Proposition: Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be a function. Then:

1. $f(x) \geq f^{**}(x)$ for all $x \in \mathbb{R}^n$.
2. If f is closed, proper and convex, then $f(x) = f^{**}(x)$.

Proof. (1) For all x and y, we have

$$f^*(y) \geq \langle x, y \rangle$$

So $f(x) \geq \langle x, y \rangle - f^*(y)$ for all x, y. (*)
Therefore, $f(x) \geq \sup\{\langle x, y \rangle - f^*(y)\} = f^{**}(x)$.
(2) By (1), we have $\text{epi}(f) \subset \text{epi}(f^{**})$. We need to show $\text{epi}(f^{**}) \subset \text{epi}(f)$.
It suffices to show that $(x, f^{**}(x)) \in \text{epi}(f)$. So suppose not.
Since $\text{epi}(f)$ is a closed convex set, $(x, f^{**}(x))$ can be strictly separated from $\text{epi}(f)$. Hence

$$\langle y, z \rangle + bs < c < \langle y, x \rangle + bf^{**}(x)$$

for some y, b, c, and for all $(z, s) \in \text{epi}(f)$.
We may assume $b \neq 0$ (If not, add $\epsilon(\overline{y}, -1)$ to (y, b), where $\overline{y} \in \text{dom} f^*$).
We must have $b < 0$. Since if $b > 0$, we have a contradiction by choosing s large.
Therefore, we further assume $b = -1$. Hence, in particular, we have

$$\langle y, z \rangle - f(z) < c < \langle y, x \rangle - f^{**}(x)$$

Then taking supremum over z, we have

$$f^*(y) + f^{**}(x) < \langle x, y \rangle$$

This is a contradiction to (*). Hence $\text{epi}(f^{**}) = \text{epi}(f)$.
Therefore, $f = f^{**}$. □