Solution to MATH3310 Midterm practice

1 Integrating factors

1. **Solution:**
   $y(x)e^x - y(0)e^0 = \int_0^x ue^u du$
   $y(x) = x - 1 + 3e^x$

2. **Solution:**
   $y(x)e^x - y(0)e^0 = \int_0^x e^u e^{-u} du$
   $y(x) = (x+1)e^{-x}$

3. **solution:**
   $x^2 y(x) - y(1) = \int_1^x 10u^3 du$
   $y(x) = \frac{5}{2}x^2 + \frac{1}{2}x^{-2}$

4. **solution:**
   roots to $r^2 + r - 6 = 0$ are $r_1 = -3, r_2 = 2$.
   Find a, b such that $y(x) = ae^{2x} + be^{-3x}$ satisfies $y(0) = 1, y(1) = 2$,
   Finally, $a = \frac{2-e^{-3}}{e^2-e^{-3}}, b = \frac{e^2-2}{e^2-e^{-3}}$

5. **solution:**
   $y(x) = ae^{\alpha x} + be^{-\alpha x} + \frac{1}{3}x^2 + \frac{7}{9}$, where $\alpha = \sqrt{1.5}$ By the two initial conditions, we have
   $a = b = -\frac{1}{9(e^{\alpha}+e^{-\alpha})}$

2 Spectral method for PDE

Solve the Following PDEs

1. $$\begin{cases} \frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty) \times (0,2\pi) \\ u(0,x) = 20 & x \in (0,2\pi) \\ u(t,0) = u(t,2\pi) = 0 & t \in [0,\infty) \end{cases}$$

solution:
$u(t,x) = \sum_{n=1} a_n e^{-n^2 t} \sin(\frac{n}{2}x)$, where $a_n = \frac{\int_0^{2\pi} 20\sin(\frac{n}{2}x)dx}{\int_0^{2\pi}\sin^2(\frac{n}{2}x)dx} = \frac{40}{n\pi}[1 - (-1)^n]$

2. $$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty)\times(0,\pi) \\ u(0,x) = \sin x & x \in (0,\pi) \\ u(t,0) = u(t,\pi) = 0 & t \in [0,\infty) \end{cases}$$

   solution:
   $u(t,x) = \sum_{n=1} a_n e^{-n^2 t}\sin(nx)$, since $u(0,x) = \sin(x)$, $u(t,x) = e^{-t}\sin(x)$

3. $$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty)\times(0,L) \\ u(0,x) = \sin\frac{2\pi}{L}x + \cos\frac{2\pi}{L}x & x \in (0,L) \\ u(t,0) = u(t,L), \quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) & t \in [0,\infty) \end{cases}$$

   solution:
   $u(t,x) = a_0 + \sum_{n=1}[a_n e^{-k(n\cdot\frac{2\pi}{L})^2 t}\sin(n\cdot\frac{2\pi}{L}x) + b_n e^{-k(n\cdot\frac{2\pi}{L})^2 t}\cos(n\cdot\frac{2\pi}{L}x)]$, since $u(0,x) = \sin\frac{2\pi}{L}x + \cos\frac{2\pi}{L}x$, then $u(t,x) = e^{-k(\frac{2\pi}{L})^2 t}(\sin\frac{2\pi}{L}x + \cos\frac{2\pi}{L}x)$

4. $$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & (t,x) \in (0,\infty)\times(0,2\pi) \\ u(0,x) = u_0(x) & x \in (0,2\pi) \\ \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,2\pi) = 0 & t \in [0,\infty) \end{cases}$$

   solution:
   $u(t,x) = a_0 + \sum_n a_n e^{-(\frac{n}{2})^2 t}\cos(\frac{n}{2}x)$
   $a_0 = \frac{\int_0^{2\pi} u_0(x)dx}{\int_0^{2\pi} 1dx}$ and $a_n = \frac{\int_0^{2\pi} u_0(x)\cos(\frac{n}{2}x)dx}{\int_0^{2\pi}\cos^2\frac{n}{2}xdx}$

3 Computing Fourier series

Compute the complex Fourier series of the following 2π-periodic functions

1. $f(x) = -|x| + \pi, -\pi \leq x \leq \pi$

   solution:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \pi$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}[\int_{-\pi}^{0} xe^{-ikx}dx + \int_{0}^{\pi}(-x)e^{-ikx}dx$$

$$= -2\int_{0}^{\pi} x\cos(kx)dx = \frac{2(1-(-1)^k)}{k^2}$$

2. $f(x) = \begin{cases} \pi & -\pi \le x \le 0 \\ x & 0 < x \le \pi \end{cases}$

solution:

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} \pi e^{-ikx}dx + \frac{1}{2\pi}\int_{0}^{\pi} xe^{-ikx}dx \\ &= \begin{cases} \frac{e^{-i\pi k}+i\pi k-1}{2\pi k^2} & \text{if } k \ne 0 \\ \frac{3\pi}{4} & \text{if } k = 0 \end{cases} \end{aligned}$$

3. $f(x) = \begin{cases} -x-\pi & -\pi \le x \le 0 \\ -x+\pi & 0 < x \le \pi \end{cases}$

solution:

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0}(-x-\pi)e^{-ikx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi}(-x+\pi)e^{-ikx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -xe^{-ikx}\,dx + \frac{1}{2\pi}\int_{-\pi}^{0} -\pi e^{-ikx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \pi e^{-ikx}\,dx \\ &= \begin{cases} -i/k & \text{if } k \ne 0 \\ 0 & \text{if } k = 0 \end{cases} \end{aligned}$$

4 Computing Fourier transform(Optional)

Compute the Fourier transforms of the following functions

1. (chanllenging) $f(x) = \frac{1}{a^2+x^2}$; (hint: you can do question 2 first, and apply Inverse Fourier transform for this question)

   solution:
   Assume $a > 0$. To solve this one, you may solve the next question first. From the next one and the inverse Fourier transform formula, we have

$$e^{-a|x|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + k^2} e^{ikx} dk$$

   Then by a simple substitution,

$$\frac{\pi}{a} e^{-a|x|} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2 + k^2} dk$$

   Hence,

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{a^2 + x^2} dx = \frac{\pi}{a} e^{-a|k|}$$

2. $f(x) = e^{-a|x|}$;

   solution:

$$\begin{aligned}
\hat{f}(k) &= \int_{\mathbb{R}} e^{-a|x|} e^{-ikx}\, dx \\
&= \int_{-\infty}^{0} e^{-a(-x)} e^{-ikx}\, dx + \int_{0}^{\infty} e^{-a(x)} e^{-ikx}\, dx \\
&= \int_{-\infty}^{0} e^{-(ik-a)x}\, dx + \int_{0}^{\infty} e^{-(ik+a)x}\, dx \\
&= \left[-\frac{e^{-(ik-a)x}}{ik-a} \right]_{-\infty}^{0} + \left[-\frac{e^{-(ik+a)x}}{ik+a} \right]_{0}^{\infty} \\
&= -\frac{1}{ik-a} + \frac{1}{ik+a} \\
&= \frac{2a}{a^2 + k^2}
\end{aligned}$$

3. $f(x) = \begin{cases} 0 & |x| > a \\ |x| & \text{otherwise} \end{cases}$.

solution:

$$\begin{aligned}
\hat{f}(k) &= \int_{\mathbb{R}} f(x)e^{-ikx}\,dx \\
&= \int_{-a}^{0} (-x)e^{-ikx}\,dx + \int_{0}^{a} xe^{-ikx}\,dx \\
&= \int_{0}^{a} xe^{ikx}\,dx + \int_{0}^{a} xe^{-ikx}\,dx \\
&= 2\int_{0}^{a} x\cos kx\,dx \\
&= 2a\frac{\sin ka}{k} + 2\frac{\cos ka}{k^2} - 2\frac{1}{k^2}
\end{aligned}$$

5 Discrete Fourier transform and numerical PDE

1. Consider the PDE: $\frac{d^2u}{dx^2} = xe^x$ for $x \in [0, 2\pi]$ with periodic boundary condition. Divide the interval $[0, 2\pi]$ using 9 points: $x_0, \ldots, x_8$. Approximate $\frac{d^2}{dx^2}$ by central difference approximation. Use the spectral method to approximate: $u_0 = u(x_0), \ldots, u_8 = u(x_8)$.

   solution:
   We divide the interval $[0, 2\pi]$ into $N = 8$ subintervals of the same size $h = \frac{\pi}{4}$, which results in 9 node points, say $x_0, \ldots, x_7$. (But under periodic boundary condition, we identify x_8 with x_0, so let's omitted x_8). Further, we take $u_0 = u(x_0), \ldots, u_7 = u(x_7)$ and use the spectral method to approximate these function values.

   Using central difference scheme, we have the following approximation.

   $$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \approx x_j e^{x_j} := f_j \quad \text{for } j = 0, \ldots, 7$$

   This results the following linear system by writing $\vec{u} = (u_0, \ldots, u_7)^T$ and $\vec{f} = (f_0, \ldots, f_7)^T$.

   $$D^2\vec{u} = \vec{f}, \quad \text{where } D^2 = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

Note that $\vec{w}_k$ for $k = 0, 1, \dots, 7$ form an eigenbasis of D^2 in $\mathbb{C}^n$ with eigenvalues $-\lambda_k^2$ respectively.

$$\vec{w}_k = (e^{ikx_0}, e^{ikx_1}, \dots, e^{ikx_7}), \quad -\lambda_k^2 := \frac{-4\sin^2\frac{kh}{2}}{h^2}$$

In other words, the discrete Fourier transform diagonalizes D^2. So, we could express $\vec{u}$ and $\vec{f}$ using $\{\vec{w}_k\}_{k=0}^7$.

$$\vec{u} = \sum_{k=0}^{7} \hat{u}_k \vec{w}_k, \quad \vec{f} = \sum_{k=0}^{7} \hat{f}_k \vec{w}_k$$

In particular, the coefficients $\hat{f}_k$ can be determined by the discrete Fourier transform.

$$\hat{f}_k = \frac{1}{8}\sum_{j=0}^{7} f_j e^{-ikx_j}, \quad \text{for } k = 0, 1, \dots, 7$$

Then putting everything back to the matrix equation $D^2\vec{u} = \vec{f}$, we have a (much) simpler system of linear equations.

$$-\lambda_k^2 \hat{u}_k = \hat{f}_k, \quad \text{for } k = 0, 1, \dots, 7$$

Note that $\hat{u}_k$ are uniquely determined for $k = 1, 2, \dots, 7$, but $\hat{u}_0$ can be arbitrary as $-\lambda_0^2 = 0$.

Let $\hat{u}_0 = c$ be arbitrary, then we have the following solution.

$$u = \sum_{k=0}^{7} \hat{u}_k e^{i\vec{kx}} = c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \sum_{k=1}^{7} \hat{u}_k e^{i\vec{kx}}, \quad \text{where } \hat{u}_k = \hat{f}_k/(-\lambda_k^2)$$

2. Let $p_1^{(0)}, \dots, p_n^{(0)}$ be n points in $\mathbb{R}^N$, given in such an order. Assume further that their mass is centered at the origin

$$\frac{1}{n}\sum_{i=1}^{n} p_i^{(0)} = 0$$

Each time we compute the midpoint of the two neighboring points,

$$p_i^{(k)} = \frac{1}{2}\left(p_i^{(k-1)} + p_{i+1}^{(k-1)}\right)$$

and we consider $p_1^{(k)}$ and $p_n^{(k)}$ to be also neighboring points. Show that these n points converge to the origin.

solution:

First, we notice that the transformation of the points is essentially the same in each dimension of $\mathbb{R}^N$. So, without loss of generality, we assume $N = 1$.

Now, we express the n points, or n real values, as a vector $\vec{p}^{(t)}$ in $\mathbb{R}^n$.

$$\vec{p}^{(t)} = (p_1^{(t)}, p_2^{(t)}, \cdots, p_n^{(t)})^T$$

Then the operation of taking mid-points can be formulated using a matrix operator.

$$\vec{p}^{(t)} = L\vec{p}^{(t-1)}, \quad \text{where } L := \frac{1}{2}\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{pmatrix}$$

Note that L is a circulant matrix, so it can be diagonalized by the discrete Fourier transform. (If you don't know the idea, you should have a look at the tutorial 4 note.)

More concretely, one could check that $\{\tilde{\omega}_k\}_{k=0}^{n-1}$ forms a basis of $\mathbb{C}^n$ and $\tilde{\omega}_k$ are eigenvectors of L with eigenvalue λ_k.

$$\tilde{\omega}_k = (e^{ikx_0}, e^{ikx_1}, \ldots, e^{ikx_{n-1}})^T, \quad \lambda_k = \frac{1 + e^{ik\frac{2\pi}{n}}}{2}$$

$$(L\tilde{\omega}_k)_j = \frac{1}{2}(e^{i(j-1)k\frac{2\pi}{n}} + e^{ijk\frac{2\pi}{n}}) = \left(\frac{1 + e^{ik\frac{2\pi}{n}}}{2}\right) e^{i(j-1)k\frac{2\pi}{n}}$$

Now, the vector $\vec{p}^{(0)}$ can now be expressed using the basis $\{\tilde{\omega}_k\}_{k=0}^{n-1}$.

$$\vec{p}^{(0)} = \sum_{k=0}^{n-1} a_k \tilde{\omega}_k$$

In particular, we have $a_0 = 0$ by the given assumption that the mass of the points is centered at the origin.

$$0 = \sum_{j=1}^{n} p_j^{(0)} = \sum_{j=1}^{n}\sum_{k=0}^{n-1} a_k e^{i(j-1)k\frac{2\pi}{n}} = \sum_{k=0}^{n-1} a_k \sum_{j=1}^{n} e^{i(j-1)k\frac{2\pi}{n}} = na_0$$

Further, we have $|\lambda_k| < 1$ for $k = 1, 2, \ldots, n-1$. Now, the convergence is obvious.

$$\vec{p}^{(t)} = L^t \sum_{k=1}^{n-1} a_k \tilde{\omega}_k = \sum_{k=1}^{n-1} a_k \lambda_k^t \tilde{\omega}_k \to \vec{0} \text{ as } t \to \infty$$

This shows that each coordinate of the points will converge to zero.

3. The auto-correlation of a vector $\mathbf{v} \in \mathbb{C}^N$ is defined to be

$$\mathbf{r} = T_{\bar{\mathbf{v}}}\mathbf{v}$$

where $r_j = \sum_{m=1}^{N} v_{j-m} v_m$ What are the discrete Fourier coefficients of $\mathbf{r}$?

solution:
We could apply the discrete Fourier transform on $\vec{r}$.

$$\begin{aligned}
\hat{r}_k &= \frac{1}{N}\sum_{j=0}^{N-1} r_j e^{-ikj\frac{2\pi}{N}} \\
&= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{m=1}^{N} v_{j-m} v_m e^{-ikj\frac{2\pi}{N}} \\
&= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{m=1}^{N} v_{j-m} e^{-ik(j-m)\frac{2\pi}{N}} v_m e^{-ikm\frac{2\pi}{N}}
\end{aligned}$$

Under periodic indexing and periodicity of exponencial function, we simply have the following.

$$\hat{r}_k = N \cdot \left[\frac{1}{N}\sum_{j=1}^{N} v_j e^{-ikj\frac{2\pi}{N}}\right]\left[\frac{1}{N}\sum_{m=1}^{N} v_m e^{-im\frac{2\pi}{N}}\right] = N \cdot [\mathrm{DFT}(v)]_k \cdot [\mathrm{DFT}(v)]_k$$

In other words, the discrete Fourier transform of the autocorrelation is just given by the entrywise product of the discrete Fourier transform of the signal and that of the complex conjugate of the signal. It also illustrates an efficient algorithm for computing autocorrelation with $\mathcal{O}(N \log N)$ (using fast Fourier transform), which is more efficient than the naive method with $\mathcal{O}(N^2)$ that computes the summation according to the definition.

6 Iterative methods (optional)

1. The Jacobi iteration for a general 2×2 matrix has

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, M = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

   Find the eigenvalues of $B = M^{-1}(M - A)$. If A is symmetric and positive definite, show that the iteration converges.
   solution:
   Obviously, we have the following.

$$B = \begin{pmatrix} 0 & -\frac{b}{a} \\ -\frac{d}{c} & 0 \end{pmatrix}$$

   So, the eigenvalue of B is given by the solutions to $\lambda^2 - \frac{bc}{ad} = 0$.

   Now, if A is symmetric, A is diagonalizable. Further, if A is positive definite, the eigenvalues of A are positive and hence the determinant of A is positive, that is, $ad - bc > 0$. In other words, we have $|\lambda|^2 = |\frac{bc}{ad}| < 1$.

   To solve $Ax = b$, the Jacobi method iterates $\vec{x} \leftarrow B\vec{x} + M^{-1}\vec{b}$. We now have $\rho(B) < 1$. Hence, the iteration converges.

2. For the Gauss-Seidel the matrices are

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

   Find the eigenvalues of $B = M^{-1}(M - A)$. Give an example of a matrix A for which the Gauss-Seidel iteration will NOT converge.
   solution: omitted

3. Decide the convergence or divergence of Jacobi and Gauss-Seidel method iterations for

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

Construct M for both methods and find the eigenvalues of $B = I - M^{-1}A$.

solution:

Let $Ax = \vec{b}$. Suppose we split A into $M + (A - M)$, that is, we have $M\vec{x} = (M - A)\vec{x} + \vec{b}$. So, an iterative scheme will iterate $\vec{x} \leftarrow M^{-1}(M - A)\vec{x} + M^{-1}\vec{b}$.

Jacobi method

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

From the characteristic polynomial of B, which is $-\lambda^3$, we see that the eigenvalues of B are all zero. Thus, the spectral radius of B is actually zero, which implies a (fast) convergence.

Gauss-Seidel method

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

From the characteristic polynomial of B, which is $-\lambda^3 + 4\lambda^2 - 4\lambda$, we see that 2 is an eigenvalue. Thus, the spectral radius is larger than 1, which means the method diverges.

7 Questions that help you understand the course materials

1. In this exercise we show how the symmetries of a function imply certain properties of its Fourier series. Let $f \in C([-\pi, \pi], \mathbb{C})$, and

$$\hat{f}(n) = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(x) e^{-inx} dx$$

(a) Show that the Fourier series of the function f can be written as

$$\hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n) + \hat{f}(-n)] \cos\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin\theta$$

(b) Show that if f is even, then $\hat{f}(n) - \hat{f}(-n) = 0$, so we get a cosine series (with possibly complex coefficients).
(c) Show that if f is odd, then $\hat{f}(n) + \hat{f}(-n) = 0$, so we get a sine series (with possibly complex coefficients).
(d) Show that $f : [-\pi, \pi] \to \mathbb{R}$, i.e. real valued, if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$. So the coefficients of cosines and sines are real. Because of this property, if f is real-valued, sometimes we call

$$\hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n) + \hat{f}(-n)] \cos\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin\theta$$

the real Fourier series, and

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

the complex Fourier series. They are seen to be equivalent expressions for $f \in C([-\pi, \pi], \mathbb{R})$.
solution:
Please refer to Assignment 2 Q3.

2. Suppose $f, g : \mathbb{R} \to \mathbb{C}$ are continuous and 2π-periodic. Then

$$\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n), \quad n \in \mathbb{Z}$$

solution: Note that the convolution for periodic functions is given by the following.

$$(f * g)(x) = \int_0^{2\pi} f(x - y) g(y)\, dy$$

Then, by definition, we have the following.

$$
\begin{aligned}
\widehat{(f*g)}(n) &= \frac{1}{2\pi}\int_{[0,2\pi]} (f*g)(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{[0,2\pi]}\int_{[0,2\pi]} f(x-y)g(y)e^{-inx}\,dy\,dx \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} g(y)e^{-iny}\int_{[0,2\pi]} f(x-y)e^{-i(x-y)}\,dx\,dy \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} g(y)e^{-iny}\,dy\int_{[0,2\pi]} f(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}(2\pi\hat{g}(n))(2\pi\hat{f}(n)) \\
&= 2\pi\hat{f}(n)\hat{g}(n)
\end{aligned}
$$

3. Show that
   (a) for a function $f : [0, 2\pi] \to \mathbb{C}$, its zero-th Fourier coefficient $\hat{f}(0)$ is the average of the function f up to dividing by 2π

   $$\hat{f}(0) = \frac{1}{2\pi}\int_{[0,2\pi]} f(x)dx$$

   (b) For a function $f : \mathbb{R} \to \mathbb{C}$, its Fourier transform evaluated at 0 is the average of the function

   $$\hat{f}(0) = \int_{\mathbb{R}} f(x)dx$$

   solution:
   The two equality follows immediately from the definitions of Fourier series and Fourier transform. More importantly, you should understand the meaning behind the zero-th Fourier coefficient and $\hat{f}(0)$.

4. What is the matrix for the central difference scheme with homogeneous Dirichlet boundary condition? Can you still diagonalize it with discrete Fourier transform?
   solution:
   Suppose we discretize the domain into N intervals of size h, so we get $N + 1$ node points, say x_0, x_1, ..., x_N. Then a homogeneous Dirichlet boundary condition means $u(x_0) = u(x_N) = 0$.

Using central difference scheme, we have

$$u''(x_j) \approx \frac{u(x_{j-1} - 2u(x_j) + u(x_{j+1})}{h^2} \quad \text{for } j = 1, \dots, N-1.$$

Hence, we have the following matrix equation.

$$\begin{pmatrix} u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx D^2 \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{N-1}) \end{pmatrix}, \text{where } D^2 = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \in \mathbf{M}^{(N-1)^2}(\mathbb{R})$$

Unfortunately, D^2 is not a circulant matrix. So, we cannot diagonalize it with discrete Fourier transform.

5. What is the matrix for the central difference scheme with homogeneous Neumann boundary condition? Can you still diagonalize it with discrete Fourier transform?
solution:
Suppose we discretize the domain into N intervals of size h, so we get $N+1$ node points, say x_0, x_1, ..., x_N. Then a homogeneous Neumann boundary condition means $u'(x_0) = u'(x_N) = 0$.

By a central difference approximation, we have

$$\frac{u(x_1) - u(x_{-1})}{2h} = 0 \quad \text{and} \quad \frac{u(x_{N+1}) - u(x_{N-1})}{2h} = 0,$$

which implies $u(x_{-1}) = u(x_1)$ and $u(x_{N+1}) = u(x_{N-1})$.

Using central difference scheme again, we now have

$$u''(x_j) \approx \frac{u(x_{j-1} - 2u(x_j) + u(x_{j+1})}{h^2} \quad \text{for } j = 0, \dots, N.$$

Hence, we have the following matrix equation.

$$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_N) \end{pmatrix} \approx D^2 \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}, \quad \text{where } D^2 = \frac{1}{h^2} \begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix} \in \mathbf{M}^{(N+1)\times(N+1)}(\mathbb{R})$$

Again, D^2 is not a circulant matrix. So, we cannot diagonalize it with discrete Fourier transform.