2425 MATH3310 Computational and Applied Mathematics

Midterm Examination

Please show all your steps, unless otherwise stated. Answer all **4** questions. The total score is **100**.

1. Consider the following ODE:

$$-u'' - 2u' + 3u = e^{-t}\cos 2t, \quad t \in \mathbb{R}$$

Follow the steps below to solve the ODE. Please show all your steps in detail.

(a) Let $v(t) = u(t)e^t$, show that $v(t)$ satisfies the following equation

$$-v'' + 4v = \cos 2t, \quad t \in \mathbb{R}$$

(b) Find the general solution to the above equation.
(Hint: A particular solution is also a trigonometric function)

(c) Solve the ODE with $u(0) = 1, u'(0) = -\frac{1}{4}$

solution:

(a) $u = ve^{-t}, u' = e^{-t}(v' - v)$, and $u'' = e^{-t}(v'' - 2v' + v)$.
So $-u'' - 2u' + 3u = e^{-t}(-v'' + 4v) = e^{-t}\cos 2t$

(b) $v(t) = Ae^{-2t} + Be^{2t} + \frac{1}{8}\cos 2t$

(c) general solution $u(t) = Ae^{-3t} + Be^{t} + \frac{1}{8}e^{-t}\cos 2t$. With $u(0) = 1, u'(0) = -\frac{1}{4}$, $A = \frac{2}{8}, B = \frac{5}{8}$

2. (a) Let $f(x)$ be a 2π-periodic function and $f(x) = |x|$ for $x \in [-\pi, \pi]$. Find the real Fourier series of $f(x)$. Please show all your steps in detail.

(b) Using (a), find the series solution to the following heat equation whose domain is $\{(x,t)\colon t > 0, x \in [-\pi, \pi]\}$.

$$\begin{cases} u_t = u_{xx} & t \geq 0, \quad x \in (-\pi, \pi) \\ u_x(-\pi, t) = u_x(\pi, t) & t > 0 \\ u(-\pi, t) = u(\pi, t) & t > 0 \\ u(x, 0) = |x| & x \in [-\pi, \pi] \end{cases}$$

(c) Using (b), find the series solution to the following heat equation whose domain is $\{(x,t)\colon t > 0, x \in [-\pi, \pi]\}$.

$$\begin{cases} u_t = u_{xx} + \sum_{n=1}^{N} \left((2n)^4 t + (2n)^2\right)\cos(2nx) + \frac{1}{2} & t \geq 0, \quad x \in (-\pi, \pi) \\ u_x(-\pi, t) = u_x(-\pi, t) & t > 0 \\ u(-\pi, t) = u(\pi, t) & t > 0 \\ u(x, 0) = |x| & x \in [-\pi, \pi] \end{cases}$$

where N is a positive integer.

solution:

(a) Suppose $f(x) = a_0 + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$,

$$\begin{aligned} a_0 &= \frac{\int_{-\pi}^{\pi} |x| dx}{\int_{-\pi}^{\pi} 1 dx} = \frac{\pi}{2} \\ a_n &= \frac{\int_{-\pi}^{\pi} |x| \cos nx dx}{\int_{-\pi}^{\pi} \cos^2 nx dx} = \frac{2\int_0^{\pi} x \cos nx dx}{\int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2} dx} = \frac{2((-1)^n - 1)}{n^2 \pi} \\ b_n &= 0 \end{aligned}$$

$|x| = \frac{\pi}{2} + \sum\limits_{n=1}^{\infty} \frac{2((-1)^n-1)}{n^2\pi} \cos nx,\ x \in [-\pi, \pi]$

(b) According to the boundary condition $u_x(-\pi, t) = u_x(\pi, t), u(-\pi, t) = u(\pi, t)$, all eigenfunctions are $\{\sin nx, \cos nx \colon n = 0, 1, 2...\}$.

Thus, suppose $u(x,t) = a_0(t) + \sum\limits_{n=1}^{\infty} [a_n(t) \cos nx + b_n(t) \sin nx]$, by $u_t = u_{xx}$, we have

$$a_0'(t) + \sum_{n=1}^{\infty} [a_n'(t) \cos nx + b_n'(t) \sin nx] = \sum_{n=1}^{\infty} \left[-n^2 a_n(t) \cos nx + -n^2 b_n(t) \sin nx\right]$$

Hence $a_n(t) = e^{-n^2 t} a_n(0), b_n(t) = e^{-n^2 t} b_n(0), a_0(t) = a_0(0)$.

By the expansion obtained in (a), $b_n(0) = 0, a_n(0) = \frac{2((-1)^n-1)}{n^2\pi}, a_0(0) = \frac{\pi}{2}$. We have

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2 \pi} e^{-n^2 t} \cos nx$$

(c) Still, suppose $u(x,t) = a_0(t) + \sum\limits_{n=1}^{\infty} [a_n(t) \cos nx + b_n(t) \sin nx]$, this time the ODE for $a_n(t)$ is that

For n even and $n \leq 2N$,

$$\begin{cases} a_n'(t) &= -n^2 a_n(t) + n^4 t + n^2 \\ a_n(0) &= \frac{2((-1)^n-1)}{n^2\pi} = 0 \end{cases}$$

For other positive n,

$$\begin{cases} a_n'(t) &= -n^2 a_n(t) \\ a_n(0) &= \frac{2((-1)^n-1)}{n^2\pi} \end{cases}$$

For $n = 0$,

$$\begin{cases} a_0'(t) &= \frac{1}{2} \\ a_0(0) &= \frac{\pi}{2} \end{cases}$$

Then for the last two cases, it's easy to obtain the answer $a_n(t) = \frac{2((-1)^n-1)}{n^2\pi}e^{-n^2t}$ and $a_0(t) = \frac{t+\pi}{2}$. For the first case,

$$\begin{aligned}
(a_n(t)e^{n^2t})' &= (n^4t+n^2)e^{n^2t} \\
a_n(t)e^{n^2t} - a_n(0) &= \int_0^t (n^4s+n^2)e^{n^2s}ds = \int_0^t (n^2s+1)e^{n^2s}d(n^2s) \\
&= \int_0^{n^2t} (u+1)e^u du = ue^u\Big|_0^{n^2t} \\
a_n(t) &= n^2t
\end{aligned}$$

3. Consider the differential equation:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume u and f are periodically extended to $\mathbb{R}$. Divide the interval $[0, 2\pi]$ into n equal portions, where $n = 2^l$ for some $l > 10$. Let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n \times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} \quad \text{and} \quad (\mathcal{D}_2\mathbf{u})_j = \frac{u(x_{j+2}) - 2u(x_j) + u(x_{j-2})}{4h^2},$$

and we can discretize the differential equation as

$$(*) \qquad a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}$$

(a) Explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in \mathbb{C}$ and $\overrightarrow{e^{ikx}} = (e^{ikx_0}, e^{ikx_1}, \ldots, e^{ikx_{N-1}})$. If $\mathbf{u}$ satisfies (*), show that

$$(a\lambda_k + b\widetilde{\lambda}_k)\hat{u}_k = \hat{f}_k \text{ for some } \lambda_k \text{ and } \widetilde{\lambda}_k,$$

for $k = 0, 1, 2, ..., n-1$. What are λ_k and $\widetilde{\lambda}_k$? Please explain your answer with details.

(c) What is the general solution of (*)? Please show and explain your answer with details.

solution:

(a) By taylor expansion,

$$u(x_{j+2}) = u(x_j) + 2hu'(x_j) + \frac{(2h)^2}{2!}u''(x_j) + \frac{(2h)^3}{3!}u'''(x_j) + O(h^4)$$

$$u(x_{j-2}) = u(x_j) - 2hu'(x_j) + \frac{(2h)^2}{2!}u''(x_j) - \frac{(2h)^3}{3!}u'''(x_j) + O(h^4)$$

Thus,

$$\frac{u(x_{j+2}) - u(x_{j-2})}{4h} = u'(x_j) + O(h^2)$$

$$\frac{u(x_{j+2}) - 2u(x_j) + u(x_{j-2})}{4h^2} = u''(x_j) + O(h^2)$$

(b) We prove that $\overrightarrow{e^{ikx}}$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$, for $k = 0, 1, ..., n-1$.

$$\mathcal{D}_1 e^{ikx_j} = \frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4h} = e^{ikx_j}\frac{e^{ik2h} - e^{-ik2h}}{4h} = \frac{i\sin(2kh)}{2h}e^{ikx_j}$$

$$\mathcal{D}_2 e^{ikx_j} = \frac{u(x_{j+2}) - 2u(x_j) + u(x_{j-2})}{4h^2} = \frac{e^{i2kh} + e^{-ik2h} - 2}{4h^2}e^{ikx_j} = \frac{-\sin^2(kh)}{h^2}e^{ikx_j}$$

Therefore

$$a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \sum_{k=0}^{n-1} a\hat{u}_k\mathcal{D}_1\overrightarrow{e^{ikx}} + b\hat{u}_k\mathcal{D}_2\overrightarrow{e^{ikx}} = \sum_{k=0}^{n-1}(a\lambda_k + b\widetilde{\lambda}_k)\hat{u}_k\overrightarrow{e^{ikx}}$$

where $\lambda_k = \frac{-\sin^2(kh)}{h^2}$ and $\widetilde{\lambda}_k = \frac{i\sin(2kh)}{2h}$. Since $\left\{\overrightarrow{e^{ikx}}\right\}$ is linearly independent, $(a\lambda_k + b\widetilde{\lambda}_k)\hat{u}_k = \hat{f}_k$

(c) $(a\lambda_k + b\widetilde{\lambda}_k) = a\frac{i\sin(2kh)}{h} + b\frac{-\sin^2(kh)}{h^2} = \sin(kh)(\frac{2ah\cdot i\cos(kh) - b\sin(kh)}{h^2})$.
$a\lambda_k + b\widetilde{\lambda}_k = 0$ iff $\sin(kh) = 0$, equivalently $\frac{2\pi}{n}k = 0$ or π. Since n is even, there are two k satisfying $a\lambda_k + b\widetilde{\lambda}_k = 0$, namely $k = 0, \frac{n}{2}$. Hence the general solution can be written as

$$u(x) = A\overrightarrow{e^{i0x}} + B\overrightarrow{e^{i\frac{n}{2}x}} + \sum_{k\neq 0,\frac{n}{2}}^{n-1}\frac{\hat{f}_k}{a\lambda_k + b\widetilde{\lambda}_k}\overrightarrow{e^{ikx}}$$

4. Let $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ and A be a circulant matrix that

$$A = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

The circular convolution of two vectors $\vec{x} = (x_0, x_1, \cdots, x_{n-1}), \vec{y} = (y_0, y_1, \cdots, y_{n-1})$ is defined as a vector $\vec{x} * \vec{y}$ such that

$$(\vec{x} * \vec{y})_j = \sum_{l=0}^{n-1} x_l y_{j-l}, \quad j = 0, ..., n-1$$

where $y_j = y_{j+n}, x_j = x_{j+n}$, for any $j \in \mathbb{Z}$. Entry-wise product of two vectors is defined as $\vec{x} \odot \vec{y} := (x_0 \cdot y_0, \cdots, x_{N-1} \cdot y_{N-1})$.

Here are definitions of discrete Fourier transform and inverse discrete Fourier transform.

$$\text{DFT}(\vec{x})(j) = \frac{1}{N}\sum_{l=0}^{N-1} x_l e^{-i\frac{2\pi}{N}jl}$$

$$\text{iDFT}(\vec{x})(j) = \sum_{l=0}^{N-1} x_l e^{+i\frac{2\pi}{N}jl}$$

(a) show that $A\vec{x} = \vec{x} * \vec{c}$

(b) Suppose $\vec{x} = \text{DFT}(\vec{f})$ and $\vec{c} = \text{DFT}(\vec{g})$ for some vectors $\vec{f}, \vec{g}$. Show that

$$A\vec{x} = \text{DFT}(f \odot g)$$

Hint: You may try to prove $\text{iDFT}(\vec{x} * \vec{c}) = \text{iDFT}(\vec{x}) \odot \text{iDFT}(\vec{c})$

solution:

(a) $(A\vec{x})(j) = x_0 c_j + x_1 c_{j-1} + \cdots + x_{n-1} c_{j+1} = \sum_{l=0}^{n-1} x_l c_{j-l}$

(b) it reduces to prove $\text{DFT}(\vec{f}) * \text{DFT}(\vec{g}) = \text{DFT}(f \odot g)$. For any fixed k,

$$\begin{aligned}
\text{DFT}(\vec{f}) * \text{DFT}(\vec{g})(k) &= \sum_{j=0}^{n-1} \hat{f}_j \hat{g}_{k-j} = \sum_{j=0}^{n-1} (\frac{1}{n}\sum_{m=0}^{n-1} f_m e^{-i\frac{2\pi}{n}jm}) \cdots (\frac{1}{n}\sum_{l=0}^{n-1} g_l e^{-i\frac{2\pi}{n}(k-j)l}) \\
&= \frac{1}{n^2}\sum_{m,l=0}^{n-1} f_m g_l e^{-i\frac{2\pi}{n}kl} \sum_{j=0}^{n-1} e^{i\frac{2\pi}{n}(l-m)j} \\
&= \frac{1}{n^2}\sum_{m,l=0}^{N-1} f_m g_l e^{-i\frac{2\pi}{n}kl} n \cdot I(m = l) \\
&= \frac{1}{n}\sum_{m=0}^{n-1} f_m g_m e^{-i\frac{2\pi}{n}km} = \text{DFT}(f \odot g)(k)
\end{aligned}$$

$$\sum_{j=0}^{n-1} e^{i\frac{2\pi}{n}(l-m)j} = \begin{cases} \sum_{j=0}^{n-1} 1 = n, & \text{if } l = m \\ \frac{1-e^{i\frac{2\pi}{n}(l-m)n}}{1-e^{i\frac{2\pi}{n}(l-m)}} = 0, & \text{if } l \neq m \end{cases}$$