TUTORIAL 3

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1 Complex Fourier series

Definition 1.1. For a 1-periodic function $f\colon \mathbb{R} \to \mathbb{C}$, we define its Fourier series as the function $\hat{f}\colon \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}dx$$

The value $\hat{f}(k)$ is sometimes called the k-th Fourier coefficient.

Proposition 1.2. For a 1-periodic function f we have

1. if $h(x) := f(x+r)$, for some $r \in \mathbb{R}$, then $\hat{h}(k) = \hat{f}(k) \cdot e^{2\pi ikr}$
2. if $h(x) := f(x) \cdot e^{2\pi imx}$ for some $m \in \mathbb{Z}$, then $\hat{h}(k) = \hat{f}(k-m)$

And like before, the value of the complex Fourier series is equal to that of the function itself under some conditions.

Theorem 1.3. For any 1 periodic piecewise smooth f, we have

$$\sum_{k=-\infty}^{+\infty} \hat{f}(k)e^{2\pi ikx} = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

Another Theorem is *Poisson summation formula*. Proof would be given in the tutorial.

Theorem 1.4. For a nice enough $f \in L^1(\mathbb{R})$,

$$\sum_{k=-\infty}^{+\infty} f(k) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)$$

Hint: apply Theorem 1.2 to $\varphi(t) = \sum_{k=-\infty}^{+\infty} f(t+k)$

Next, we extend our definition of the Fourier series to functions whose period is not necessarily 1.

Definition 1.5. For a function $f\colon \mathbb{R} \to \mathbb{C}$ with some period $\lambda > 0$, we define its Fourier series as $\hat{F}\colon \frac{1}{\lambda}\mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(k) = \frac{1}{\lambda} \int_0^{\lambda} f(x) e^{-2\pi i k x} dx$$

This definition is derived by transforming a L periodic function to a 1-periodic function by scaling $g(x) := f(\lambda x)$. And correspondingly, we have

Theorem 1.6.

$$f(x) = \sum_{k \in \frac{1}{\lambda}\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

Theorem 1.7. For any $\lambda > 0$ and any nice enough function f,

$$\sum_{k \in \lambda\mathbb{Z}} f(x) = \frac{1}{\lambda} \sum_{y \in \frac{1}{\lambda}\mathbb{Z}} \hat{f}(y)$$

Hint: apply Theorem 1.6 to $\varphi(x) = \sum_{j=-\infty}^{+\infty} f(x + \lambda j)$

Here are some questions.

Question 1.1. Let $f(x) = e^{-\pi x^2}$, we have for any $\lambda > 0$,

$$\sum_{k=-\infty}^{+\infty} e^{-\pi(\lambda k)^2} = \frac{1}{\lambda} \sum_{k=-\infty}^{\infty} e^{-\pi(\frac{k}{\lambda})^2}$$

2 Fourier transform in general case

2.1 n-dimensional case

In this subsection, we extend the definition of the Fourier series to the n-dimensional case. We consider the Fourier series of functions on $\mathbb{R}^n$ that are $\mathbb{Z}^n$-periodic, namely functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that $f(x+y) = f(x)$ for any $x \in \mathbb{R}^n, y \in \mathbb{Z}^n$.

Definition 2.1. For a $\mathbb{Z}^n$-periodic function f, its Fourier series $\hat{f}\colon \mathbb{Z}^n \to \mathbb{C}$ is given by

$$\hat{f}(y) = \hat{f}(y_1, ..., y_n) = \int_{[0,1)^n} f(x)e^{-2\pi i\langle x,y\rangle}dx = \int_{[0,1)^n} f(x_1, ..., x_n)e^{-2\pi i\sum x_i y_i}dx1...dx_n$$

Similar theorems hold for $\mathbb{Z}^n$-periodic functions.

Theorem 2.2. For a nice enough f we have that for all x

$$f(x) = \sum_{y\in\mathbb{Z}^n} \hat{f}(y)e^{2\pi i\langle x,y\rangle}$$

Theorem 2.3. $\sum_{x\in\mathbb{Z}^n} f(x) = \sum_{y\in\mathbb{Z}^n} \hat{f}(x)$

2.2 Fourier transform for general functions

Above are Fourier transform for periodic functions, for completeness and your interest, here we introduce the extension of Fourier transform to the general case that $f \in L^1(\mathbb{R})$ and moreover $f \in L^1(\mathbb{R}^n)$.

Definition 2.4. For a function $f \in L^1(\mathbb{R})$, its Fourier transform is $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx$$

and convolution of $f, g \in L^1(\mathbb{R})$ is $f * g\colon \mathbb{R} \to \mathbb{C}$ given by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

And several important properties of the Fourier transform are listed below.

Proposition 2.5. For $f, g \in L^1(\mathbb{R}), x, y, z \in \mathbb{R}$,

1. if $\bar{f}$ is the complex conjugate of f, then $\hat{\bar{f}}(y) = \overline{\hat{f}(-y)}$
2. if $h(x) := f(x+z)$, then $\hat{h}(y) = \hat{f}(y) \cdot e^{2\pi i z y}$
3. if $h(x) := e^{2\pi i z x} f(x)$, then $\hat{h}(y) = \hat{f}(y-z)$
4. $\forall \lambda > 0$, define $h(x) := f(\lambda x)$, then $\hat{h}(y) = \frac{1}{\lambda} \hat{f}(\frac{y}{\lambda})$
5. $\widehat{f * g} = \hat{f} \cdot \hat{g}$ and $\widehat{f \cdot g} = \hat{f} \cdot \hat{g}$
6. if $h(x) = f'(x) \in L^1(\mathbb{R})$, then $\hat{h}(y) = 2\pi i y \hat{f}(y)$

Just like the periodic case, f and $\hat{f}$ have some connections.

Theorem 2.6. Suppose $f, f' \in L^1(\mathbb{R})$ and f is continuous, then

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(y) e^{2\pi i x y} dy$$

Naturally, we can extend the definition to the n-dimensional case. Denote $L^1(\mathbb{R}^n)$ as the set of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_{\mathbb{R}^n} |f(x_1, ..., x_n)| dx_1 ... dx_n < +\infty$.

Definition 2.7. For $f \in L^1(\mathbb{R}^n)$, define $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx$$

and we have similar properties and an inversion formula related to the multidimensional Fourier transform as Proposition 2.5, which are omitted here.

3 Discrete Fourier transform

Let $f \in L^2[0, 2\pi]$ be 2π periodic and continuous.

3.1 trapezoidal rule

In before, we define the k-th Fourier coefficient as an integral. But in reality, usually, we only have observations of f at certain equally spaced points $\left\{x_j = j\frac{2\pi}{N} : j = 0, ..., N\right\}$ instead of knowing the analytic formula of f. Using the trapezoidal rule, we have an approximation of $\hat{f}(k)$, denoted by $\hat{f}_k$

$$\begin{aligned} \hat{f}_k &= \frac{h}{2\pi} \left(\frac{f_0}{2} + \frac{f_N}{2} + \sum_{j=1}^{N-1} f_j e^{-ikx_j} \right) \\ &\approx \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \end{aligned}$$

Here $f_j = f(x_j)$ and $h = \frac{2\pi}{N} = x_{j+1} - x_j$. Since f is assumed to be periodic, $f_N = f_0$, so we define **discrete Fourier transform** by

Definition 3.1. $\{f_j\}_{j=0}^{N-1} \to \left\{\hat{f}_k\right\}_{k=0}^{K-1}, \quad \hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ik\cdot x_j}$

Usually, we choose K to be no less than N. (why?)

Question 3.1. Prove that only when $K \geqslant N$, we can recover $\{f_j\}_{j=0}^{N-1}$ from $\left\{\hat{f}_k\right\}_{k=0}^{K-1}$

In the following, we always assume $K = N$. Note that if f is real then we may have

$$\hat{F}_{N-1}(x) = \sum_{k=0}^{N-1} \hat{f}_k e^{ikx} = A_0 + \sum_{k=1}^{N-1} A_k \cos(kx) + \sum_{k=1}^{N-1} B_k \sin(kx) \tag{1}$$

also as a trigonometric polynomial approximating $f(x)$. And in the lecture, we know $\hat{F}_m(x_j) = f(x_j), j = 0, ..., N-1$. However, note that **this DFT approximation 1 is not the same as the partial sum of real Fourier series!!**. On the other hand, from the lecture we know the order $N-1$ partial sum of real Fourier series is the best approximation to $f(x)$ in the L^2 sense among the family of trigonometric polynomials of degree $N-1$. Is there any similar meaning for this $\hat{F}_{N-1}$? The answer is yes.

3.2 discrete least squares

Let $x_0, ..., x_N$ be equally spaced points in $[0, 2\pi]$, *discrete inner product* for $f, g \in L^2[0, 2\pi]$ is defined by

$$\langle f, g\rangle_d := \sum_{j=0}^{N-1} f(x_j)\overline{g(x_j)}$$

and a discrete L^2−norm is $\|f\|_{2,d}^2 = \langle f, f\rangle_d$. Remarkably, $\left\{e^{ikx}\right\}$ is orthogonal in the discrete inner product $\langle , \rangle_d$.

Question 3.2. Prove that for integers j, k we have

$$\langle e^{ijx}, e^{ikx}\rangle_d = \begin{cases} 0 \text{ if } j \neq k \\ N \text{ if } j = k \end{cases}$$

And denote by $\mathcal{A}$ the set of functions that can be written as $\sum_{k=0}^{N-1} a_k e^{ikx}$ for some constants a_k. Then we have

Proposition 3.2. $\hat{F}_{N-1} = \min_{g\in\mathcal{A}} \|f - g\|_{2,d}$

3.3 Convolution

In before, we have learned the convolution of two periodic functions or more generally two functions defined on $\mathbb{R}$. And we know the Fourier transform of a convolution is the product of the transforms. Similar properties hold in the discrete case. Before talking about it, we need to define ***discrete circular convolution***.

Definition 3.3. The discrete circular convolution of data vectors $(f_0, ..., f_{N-1})$ and $(g_0, ..., g_{N-1})$, presumably sampled from the periodic data as in the DFT, is defined by

$$(f * g)_j = \sum_{l=0}^{N-1} f_l g_{j-l}, \quad j = 0, ..., N-1$$

where the indices are 'mod N' (so they wrap around, e.g. N become 0, $N+1$ becomes 1 and so on).

Proposition 3.4. Let $\text{DFT}(f) = (\hat{f}_0, ..., \hat{f}_{N-1})$, we have

$$\text{DFT}(f * g) = (\hat{f}_0 \cdot \hat{g}_0, ..., \hat{f}_{N-1} \cdot \hat{g}_{N-1})$$