Math3310 Tutorial 2

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1 Fourier series and Differential Equations

1.1 Convergence, differentiation and integration

An essential assumption for computing the Fourier series of a function is that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nx) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nx)$$

But is this equation valid for all functions? Here is a counter-example, Suppose $f(x)$ is a 2π-periodic function and

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

Compute the value of its Fourier series at $x = 0$ and $x = \pi$ and compare with the counterpart of $f(x)$.

Theorem 1 *Suppose $f(t)$ is a 2L-periodic piecewise smooth function, then*

$$\frac{f(t-) + f(t+)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nt) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nt)$$

so the Fourier series converge to $f(t)$ at each continuous point.

Moreover, its antiderivative can be obtained by integration term by term,

$$F(t) = a_0 t + C + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin(\frac{n\pi}{L}t) - \frac{b_n L}{n\pi} \cos(\frac{n\pi}{L}t)$$

where $F'(t) = f(t)$ and C is an arbitrary constant.

Furthermore, if $f'(t)$ is piecewise smooth, then its derivative can be obtained by differentiating term by term.

$$f'(t) = \sum_{n=1}^{\infty} \frac{-a_n n\pi}{L} \sin(\frac{n\pi}{L}t) + \sum_{n=1}^{\infty} \frac{b_n n\pi}{L} \cos(\frac{n\pi}{L}t)$$

In addition, we may connect the Fourier series with what has been taught in linear algebra. For the conventional vector space we studied in the linear algebra course, we usually write a vector as a linear combination of an orthonormal basis $\{e_j\}$, $v = \sum_{j=1}^{n} a_j e_j$, and we have $\langle x, x \rangle = \sum_{j=1}^{n} a_j^2 \langle e_j, e_j \rangle = \sum_{j=1}^{n} a_j^2$

The spirit of Fourier series is exactly the same, $f(t)$ is a vector in the vector space of $2L$ -periodic functions, and $\left\{\cos(\frac{\pi}{L}nx), \sin(\frac{\pi}{L}nx)\right\}$ is an orthogonal basis, and most of the time,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nt) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nt)$$

this equation is valid. but two differences are that

1. this sum is infinite
2. $\left\{\cos(\frac{\pi}{L}nt), \sin(\frac{\pi}{L}nt)\right\}$ is orthogonal not orthonormal.

However, there is still an identity for a class of "good" functions.

Theorem 2 *(Parseval's Identity) Suppose $f(t)$ is a square-integrable function, i.e.*

$$\int_{-L}^{L} (f(t))^2 dt < +\infty$$

we have

$$\int_{-L}^{L}(f(t))^2dt = \langle f, f\rangle_1$$

$$= \langle a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nx) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nx), a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{L}nx) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi}{L}nx)\rangle_1$$

$$= 2L \cdot a_0^2 + L \cdot \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

1.2 infinite sum

Many seemingly difficult infinite sum equations can be proved by computing its Fourier series and substituting suitable points into it, or directly applying Parseval's Identity. For example, given a 2π-periodic $f(x)$ defined for $-\pi \leq x \leq \pi$ by $f(x) = |x|$. Its Fourier series is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{kodd}^{\infty}\frac{\cos(kx)}{k^2}$$

x = 0, we have

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

With Parseval's Identity, we have another infinite sum formula.

$$\begin{aligned}\frac{2\pi^3}{3} &= \int_{-\pi}^{\pi} x^2 dx \\ &= 2\pi \cdot (\frac{\pi}{2})^2 + \pi \cdot (\sum_{k \quad odd}\frac{4}{\pi k^2})^2 \\ &= \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\end{aligned} \tag{1}$$

Therefore,

$$\frac{\pi^4}{96} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$

Here are some exercises.

1. $f(x)$ is an 1- periodic function defined for $-\frac{1}{2} \leq |x| \leq \frac{1}{2}$ by $f(x) = x^2$. Find its Fourier series and prove the following three infinite sum formulas.
   (a) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$
   (b) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$
   (c) $\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}$
2. $f(x)$ is a 2π- periodic function defined for $-\pi \leq |x| \leq \pi$ by

$$f(x) = \begin{cases} 0, \text{ if } -\pi < x < 0 \\ \sin(x), \text{ if } 0 \leq x \leq \pi \end{cases}$$

   Find the Fourier series and prove the following infinite sum formula.
   (a) $\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$
   (b) $\frac{(\pi^2-8)}{16} = \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}$

1.3 Application to Solving Differential Equations

1.3.1 inhomogeneous ODE

In the tutorial 1, we mentioned two methods for solving inhomogeneous second-ordernd order system in the form:

$$ax'' + bx' + cx = f(t)$$

which are by guessing or by variation of parameters. But these two methods have their own drawbacks. If $f(t)$ is not a common function, like triangonometric function or polynomials, it's not easy to give a guess. As for the formula derived by variation of parameters, sometimes it is tedious to calculate the integral. Here we introduce another method for a special case of inhomogeneous ODE that $f(t)$ is a periodic function. In such a case, we may obtain the special solution by Fourier expansion.

Suppose $f(t)$ is a $2L$-periodic function, its Fourier series is like

$$f(t) = c_0 + \sum_n c_n \cos(\frac{n\pi}{L}t) + \sum_n d_n \sin(\frac{n\pi}{L}t)$$

it's natural to guess that a special solution x_{sp} is also a $2L$-periodic function and we suppose its Fourier series is also in the form:

$$x_{sp}(t) = a_0 + \sum_n a_n \cos(\frac{n\pi}{L}t) + \sum_n b_n \sin(\frac{n\pi}{L}t)$$

By differentiating termwisely, we would get linear systems by comparing coefficients of eigenfunctions on both sides.

$$\begin{cases} c \cdot c_0 = a_0 \\ (c - a \cdot (\frac{n\pi}{L})^2) \cdot a_n + b \cdot \frac{n\pi}{L} \cdot b_n = c_n, \text{ for } n \geq 1 \\ (c - a \cdot (\frac{n\pi}{L})^2) \cdot b_n - b \cdot \frac{n\pi}{L} \cdot a_n = d_n, \text{ for } n \geq 1 \end{cases}$$

Don't forget the part of solutions $v_1(t), v_2(t)$ to homogeneous ODE $ax'' + bx' + cx = 0$, and the general solution to the inhomogeneous ODE is $x(t) = \alpha_1 v_1(t) + \alpha_2 v_2(t) + a_0 + \sum_n a_n \cos(\frac{n\pi}{L}t) + \sum_n b_n \sin(\frac{n\pi}{L}t)$.

However, in some very special cases, the forms of v_1 and v_2 concide with the triagonometric functions in the x_{sp}. In this case, above system would involve terms related to α_1 and α_2, which results that there would be not only one set of a_n, b_n satisfying the system. For example,

$$2x'' + 18\pi^2 x = \sum_{n\ odd} \frac{4}{\pi n} \sin(n\pi t)$$

the general solution to this ode, following steps taught before, is $x(t) = \alpha_1 \cos(3\pi t) + \alpha_2 \sin(3\pi t) + x_{sp}(t)$ but further, like before, we expand x_{sp} to be $\sum_n b_n \sin(n\pi t)$, the term $\alpha_2 \sin(3\pi t)$ would appear in the system obtained by comparing the coefficients of sine functions.

The strategy is to modify the form of special solution a bit. We pull out the $\sin(3\pi t)$ term in the x_{sp} and multiply by t. Next, for symmetry, we add a cosine term, so the new form of x_{sp} is

$$x_{sp}(t) = a_3 t \cos(3\pi t) + b_3 t \sin(3\pi t) + \sum_{n\ odd, n\neq 3} b_n \sin(n\pi t)$$

Then,

$$\begin{aligned} 2x''_{sp} + 18\pi^2 x_{sp} &= (-12a_3\pi - 18\pi^2 a_3 t + 18\pi^2 a_3 t)\cos(3\pi t) + (12b_3\pi - 18\pi^2 b_3 t + 18\pi^2 b_3 t)\sin(3\pi t) \\ &\quad + \sum_{n\ odd, n\neq 3} (-2n^2\pi^2 b_n + 18\pi^2 b_n)\sin(n\pi t) \\ &= \sum_{n\ odd} \frac{4}{\pi n} \sin(n\pi t) \end{aligned} \tag{2}$$

which again has a unique solution set $\{a_n, b_n\}$.

Here are some exercises, note that in these exercises, f is a 2π periodic function and its formula given below is defined for $-\pi \le |x| \le \pi$.

1. $x'' + 4x = \cos(t) + \sin(t), x(0) = 1, x'(0) = 1$
2. $x'' + 2x' + x = \cos(t), x(0) = 1, x(\pi) = 1$
3. general solution to $x'' + x = t$
4. general solution to $x'' + 4x = t$

1.3.2 Separation of Variables and Heat Equation

A one-dimensional heat equation for $u(x,t)$ is like

$$u_t = ku_{xx}$$

where k is a positive constant. Usually, x is restricted in an interval $[0, L]$ and $t > 0$. For the heat equation, we must have boundary conditions, such as

$$u(0,t) = 0, u(L,t) = 0$$

or

$$u_x(0,t) = 0, u_x(L,t) = 0$$

We also need an initial condition — the temperature distribution at $t = 0$

$$u(x,0) = f(x)$$

In the class, a method *separation of variable* is introduced to solve this kind of differential equation. its idea is to represent $u(x,t)$ as a linear combination of eigenfunctions and these eigenfunctions can be written as products of functions of only one variable, that is

$$u(x,t) = constant + \sum_n a_n X_n(x) T_n(t)$$

in which, forms of X_i and T_i are determined by the differential equation $u_t = ku_{xx}$ itself as well as the interval $[0, L]$. The boundary condition helps you filter out which $a_n = 0$. Last, the initial condition determines the exact value of nonzero a_n. Here is an example,

$$\begin{cases} u_t = 2u_{xx}, \quad t > 0, 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x(1-x), \quad 0 < x < 1 \end{cases} \tag{1}$$

We plug $u_0(x,t) = X(x)T(t)$ into the heat equation, we have

$$X(x)T'(t) = 2X''(x)T(t)$$

we rewrite as

$$\frac{T'(t)}{2T(t)} = \frac{X''(x)}{X(x)}$$

Since the left hand side doesn't depend on x and the right hand side doesn't depend on t, each side must be a constant, say $-\lambda$ ($\lambda \ge 0$ and we have

$$\frac{T'(t)}{2T(t)} = -\lambda = \frac{X''(x)}{X(x)}$$

with solutions $T(t) = e^{-2\lambda t}$ and $X(x) = \sin(\sqrt{\lambda}x)$ or $\cos(\sqrt{\lambda}x)$.

How to choose λ? Remember that we have the boundary conditions $u(0,t) = u(1,t) = 0$. So we are solving the eigenvalue problem

$$X'' + \lambda X = 0, X(0) = X(1) = 0$$

We previously has proved that its eigenfunctions are $\sin(n\pi x)$ with eigenvalues $\lambda_n = n^2\pi^2$. Therefore, here λ should be $n^2\pi^2$. and correspondingly, we obtain a family of eigenfunctions for the differential equation (1), that is $\left\{e^{-2n^2\pi^2 t} \cdot \sin(n\pi x); n \in \mathbb{N}\right\}$ and we may assume the solution $u(x,t) = b_0 + \sum_n b_n e^{-2n^2\pi^2 t} \cdot \sin(n\pi x)$.

Last, given that $u(x,0) = x(1-x) = \sum\limits_{n\ odd} \frac{80}{n^3\pi^3} \sin(n\pi x)$, we have

$$u(x,t) = \sum_{n\ odd} \frac{80}{n^3\pi^3} \sin(n\pi x) e^{-2n^2\pi^2 t}$$

Here are some exercises,

1. solve the differential equation

$$\begin{cases} u_t = 2u_{xx}, \quad t > 0, 0 \le x \le 1 \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = x(1-x), \quad 0 < x < 1 \end{cases}$$

2. solve the differential equation

$$\begin{cases} u_t = 3u_{xx}, \quad t > 0, 0 \le x \le \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 5\sin(x) + 2\sin(5x), \quad 0 < x < \pi \end{cases}$$

3. (challenging) solve

$$\begin{cases} u_t = u_{xx}, \quad t > 0, 0 \le x \le \pi \\ u(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = 5\sin(\frac{5}{2}x), \quad 0 < x < \pi \end{cases}$$

Hint: the essential step is to select suitable λ_n according to the boundary conditions. Here, λ_n should be $\frac{2n+1}{2}\pi$

4. (challenging) In above, why don't we consider the case that $\lambda = 0$? Namely, $u_t = u_{xx} = 0$. For this case, it has a solution $u_0(x,t) = ax+b$. That's because both the boundary conditions are zero valued. When we have nontrival constant boundary values, such u_0 would play an important role. Solve

$$\begin{cases} u_t = a^2u_{xx}, \quad t > 0, 0 \le x \le \pi \\ u(0,t) = T_1, u(\pi,t) = T_2 \\ u(x,0) = f(x) \quad 0 < x < \pi \end{cases}$$

coefficients of the series solution can be written as an integral of $f(x)$. Hint: you may try to decompose the problem to two subproblems as we did in the next section.

1.3.3 wave equation

A one-dimensional wave equation is of the form:

$$u_{tt} = a^2 u_{xx}$$

Still we should be given two boundary conditions, like $u(0,t) = u(L,t) = 0$. But this time, we need two initial conditions to ensure uniqueness of the solution. (that's because there are two derivatives along the t direction). These conditions are always imposed in the way:

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

And again, we can use the method *separation of variable* to solve it. Like before, we have $\frac{T''(t)}{a^2T(t)} = -\lambda = \frac{X''(x)}{X(x)}$ for some $\lambda \geq 0$. The general solutions for $T(t)$ and $X(x)$ are $A\cos(a\sqrt{\lambda}t) + B\sin(a\sqrt{\lambda}t)$ and $C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$, respectively. The boundary condition help you determine the value of λ and what trigonometric functions should be left. And the initial condition plays the similar role. To make the life easier, we usually decompose the system

$$\begin{cases} u_{tt} = a^2 u_{xx}, & t > 0, 0 \leq x \leq L \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), & 0 < x < L \\ u_t(x,0) = g(x), & 0 < x < L \end{cases} \tag{2}$$

into two questions.

$$\begin{cases} y_{tt} = a^2 y_{xx}, & t > 0, 0 \leq x \leq L \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0, & 0 < x < L \\ y_t(x,0) = g(x), & 0 < x < L \end{cases} \qquad \begin{cases} z_{tt} = a^2 z_{xx}, & t > 0, 0 \leq x \leq L \\ z(0,t) = z(L,t) = 0 \\ z(x,0) = f(x), & 0 < x < L \\ z_t(x,0) = 0, & 0 < x < L \end{cases}$$

Then $u(x,t) = y(x,t) + z(x,t)$. Such a decomposition helps a lot, because each subsystem has a zero valued initial condition which helps select what kind of eigenfunction for $T(t)$ is needed, and another nontrival initial condition is to help compute the coefficients of filtered eigenfunctions.

For example, for the first system of $y(x,t)$, by the differential equation and the boundary condition, we have eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, eigenfunctions for $X(x)$ being $\sin(\frac{n\pi}{L}x)$, and eigenfunctions for $T(t)$ being $\left\{\sin(\frac{an\pi}{L}t), \cos(\frac{an\pi}{L}t)\right\}$. The boundary condition $y(x,0) = 0$ implies that $T(t) = \sum_n b_n \sin(\frac{an\pi}{L}t)$ thus $y(x,t) = \sum_n b_n \sin(\frac{an\pi}{L}t)\sin(\frac{n\pi}{L}x)$. Lastly, by differentiating termwisely, $y_t(x,0) = \sum_n a \cdot b_n \frac{n\pi}{L}\sin(\frac{n\pi}{L}x)$ and values of b_n's are obtained by writing $g(x) = \sum_n g_n \sin(\frac{n\pi}{L}x)$ and $b_n = \frac{L}{n\pi a} g_n$.

Same operation can be done on the system for $z(x,t)$, and the only difference is that this time, the filtered eigenfunctions for $T(t)$ are $\left\{\cos(\frac{an\pi}{L}t); n \in \mathbf{N}\right\}$ and $z(x,t) = \sum_{n=1}^{\infty} a_n \cos(\frac{an\pi}{L}t)\sin(\frac{n\pi}{L}x)$.

Theorem 3 *For the equation (2), suppose* $f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x)$ *and* $g(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$, *then the solution*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} b_n \sin(\frac{n\pi a}{L}t)\sin(\frac{n\pi}{L}x) + \sum_{n=1}^{\infty} c_n \cos(\frac{n\pi a}{L}t)\sin(\frac{n\pi}{L}x)$$

Here are some exercises.

1. solve

$$\begin{cases} u_{tt} = 2u_{xx}, & t > 0, 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x, & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{cases}$$

2. solve

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(x), & 0 < x < \pi \\ u_t(x,0) = \sin(x) & 0 < x < \pi \end{cases}$$

3. When $a = 0$, the eigenfunctions for $T(t)$ are t and constant functions. Solve

$$\begin{cases} u_{tt} = 0, \quad t > 0, 0 \le x \le \pi \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(2x), \quad 0 < x < \pi \\ u_t(x,0) = \sin(x) \quad 0 < x < \pi \end{cases}$$

4. (challenging) Could you follow the same idea to obtain a series solution to the following problem?

$$\begin{cases} u_{tt} = a^2 u_{xx} - k u_t, \quad t > 0, 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x), \quad 0 < x < 1 \\ u_t(x,0) = 0 \quad 0 < x < 1 \end{cases}$$

where $0 < k < 2\pi a$. Any coefficient in the series should be expressed in an integral of $f(x)$.