THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2024-2025
Assignment 1 Suggested Solution

1. Solve the following PDE using the spectral method:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in [0,1] \times (-\infty, \infty) \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = 100, \text{ for } x \in (0,1) \end{cases}$$

(Hint: Odd extension may help.)

Solution: By observation, we extend the PDE into the following form

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in [0,1] \times (-\infty, \infty) \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = 100, \text{ for } x \in (0,1) \\ u(x,0) = -100, \text{ for } x \in (-1,0) \end{cases}$$

Let $u(x,t) = a_0(t) + \sum_{n=1}^{\infty}(a_n(t)cos(\pi nx) + b_n(t)sin(\pi nx))$, then

$$u_t(x,t) = a_0'(t) + \sum_{n=1}^{\infty}(a_n'(t)cos(\pi nx) + b_n'(t)sin(\pi nx))$$
$$u_{xx}(x,t) = \sum_{n=1}^{\infty}((-\pi^2 n^2)a_n(t)cos(\pi nx) + (-\pi^2 n^2)b_n(t)sin(\pi nx))$$

Comparing the two equations, we have

$$a_0'(t) = 0$$
$$a_n'(t) = (-\pi^2 n^2)a_n(t)$$
$$b_n'(t) = (-\pi^2 n^2)b_n(t)$$

Using integrating factor method to solve the above equations, we have

$$a_0(t) = C_0$$
$$a_n(t) = C_n^a e^{-\pi^2 n^2 t}$$
$$b_n(t) = C_n^b e^{-\pi^2 n^2 t}$$

Considering the case $t = 0$, we have $u(0,0) = u(0,1) = 0$, $u(x,0) = 100, x \in (0,1)$, and $u(x,0) = -100, x \in (-1,0)$, and

$$u(x,0) = C_0 + \sum_{n=1}^{\infty}(C_n^a cos(\pi nx) + C_n^b sin(\pi nx))$$

then

$$
\begin{aligned}
C_0 &= \frac{1}{2}\int_{-1}^{1} u(x,0)dx \\
&= 0 \\
C_n^a &= \pi \cdot \frac{1}{\pi}(\int_{-1}^{0} -100cos(\pi nx)dx + \int_{0}^{1} 100cos(\pi nx)dx) \\
&= 0 \\
C_n^b &= \pi \cdot \frac{1}{\pi}(\int_{-1}^{0} -100sin(\pi nx)dx + \int_{0}^{1} 100sin(\pi nx)dx) \\
&= \begin{cases} \frac{400}{\pi n}, & \text{n is odd} \\ 0, & \text{n is even} \end{cases}
\end{aligned}
$$

Therefore,

$$
u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-\pi^2(2n-1)^2 t} sin(\pi(2n-1)x)
$$

2. A discrete complex-valued function f can be represented by a vector $(f_0, f_1, \dots, f_{n-1})^T$. Consider a matrix M where the entry in the j-th row and k-th column is given by $M_{jk} = e^{i\frac{2jk\pi}{n}}$.

   Please express the function f as a linear combination of the column vectors of M. In other words, you need to determine the coefficients for this linear combination.

   Solution:

   Considering the matrix multiplication M^*M, we notice that

$$
(M^*M)_{ij} = \sum_{k=0}^{n-1} e^{-i\frac{2ik\pi}{n}} e^{i\frac{2kj\pi}{n}} = \sum_{k=0}^{n-1} e^{i\frac{2\pi k(j-i)}{n}} = \begin{cases} N, & j = i \\ 0, & j \neq i \end{cases}
$$

   Therefore, $M^{-1} = \frac{1}{N}M^*$. Suppose we have a vector such that $f = Mc$, then $c = \frac{1}{N}M^*f$. More specifically,

$$
c_k = \frac{1}{n}\sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}}
$$

3. Let $f(x)$ be a 2π-periodic complex-valued function and $\int_0^{2\pi} |f(x)|^2 dx < \infty$. Its complex Fourier coefficient is computed by $\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$ and complex Fourier series is

$$
\mathcal{F}(f)(x) := \sum_{k=-\infty}^{+\infty} \hat{f}_k e^{ikx}
$$

   and the truncated version is

$$
\mathcal{F}_N(f)(x) := \sum_{k=-N}^{N} \hat{f}_k e^{ikx}
$$

   Recall its real Fourier series is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$.Prove that

   (a) $\hat{f}_k = \frac{a_k - ib_k}{2}$, if $k \geq 1$

   (b) $\hat{f}_k = \frac{a_{-k} + ib_{-k}}{2}$, if $k \leq -1$

   (c) If $f(x)$ is real-valued, $a_k = 2\mathbf{Re}(\hat{f}_k)$ and $b_k = -2\mathbf{Im}(\hat{f}_k)$ for $k \geq 1$

   Solution:

(a) For $k \geq 1$, $\hat{f}_k = \frac{1}{2} \cdot \frac{1}{\pi} \int_0^{2\pi} f(x)(\cos(kx) - i\sin(kx))dx = \frac{1}{2}(a_k - ib_k)$

(b) For $k \leq -1$,

$$
\begin{aligned}
, \hat{f}_k &= \frac{1}{2} \cdot \frac{1}{\pi} \int_0^{2\pi} f(x)(cos(kx) - isin(kx))dx \\
&= \frac{1}{2} \cdot \frac{1}{\pi} \int_0^{2\pi} f(x)(cos(-kx) + isin(-kx))dx \\
&= \frac{1}{2}(a_{-k} + ib_{-k})
\end{aligned}
$$

(c) because if f is real-valued, then a_k and b_k are real numbers.

4. Given a positive even integer N, let $E_k(x) = e^{ikx}$ for $k \geq 0$ and $x_j = j\frac{2\pi}{N}$ for $0 \leq j \leq N - 1$. Since $E_k(x_j) = E_{k+N}(x_j)$, we can do discrete Fourier transform with the set of functions $\left\{E_k(x) \colon k = -\frac{N}{2}+1, -\frac{N}{2}+2, ..., \frac{N}{2}\right\}$. For symmetry, we would like to do discrete Fourier transform with $E = \left\{E_k(x) \colon k = -\frac{N}{2}, -\frac{N}{2}+1, ..., \frac{N}{2}-1, \frac{N}{2}\right\}$ and updated computing rule is

$$\hat{f}_k = \frac{1}{a_k} \cdot \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) e^{-ikx_j}, \text{ for } k = -\tfrac{N}{2}, ..., \tfrac{N}{2}$$

where $a_k = 2$ if $k = \pm\frac{N}{2}$ otherwise 1. And Its inverse Discrete Fourier transform is given by

$$\left(I_N(f)\right)(x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{f}_k e^{ikx}$$

(a) Let $\tilde{f}_k = \frac{1}{N} \sum\limits_{j=0}^{N-1} f(x_j) e^{-ikx_j}$, for $k = 0, ..., N-1$. Prove that

i. $\hat{f}_k = \tilde{f}_{k+N}$, for $k = -\frac{N}{2}+1, ..., -1$

ii. $\hat{f}_{\pm\frac{N}{2}} = \frac{1}{2}\tilde{f}_{\frac{N}{2}}$

iii. $\sum_{k=0}^{N-1} |\tilde{f}_k|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |f_k|^2$

(b) Prove that

$$\left(I_N(f)\right)(x) = \sum_{j=0}^{N-1} f(x_j) g_j(x) \tag{1}$$

where

$$g_j(x) = \frac{1}{N}\sin(N\frac{x - x_j}{2})\cot(\frac{x - x_j}{2})$$

and $g_j(x_k) = 1$ if $j = k$ otherwise 0.

(c) By using the nodal basis representation (1), we can compute the derivative of $f(x)$ by $f^{(m)}(x) \approx \left(I_N(f)\right)^{(m)}(x)$. Prove that

Let $\mathbf{f}_N = \left(f(x_0), ..., f(x_{N-1})\right)^T$ and $\mathbf{f}_N^{(m)} = \left(f^{(m)}(x_0), ..., f^{(m)}(x_{N-1})\right)^T$, then $\mathbf{f}_N^{(m)} = D^m \mathbf{f}_N$ for some matrix D^m. In particular,

$$D^1(k, j) = g_j'(x_k) = \begin{cases} \frac{(-1)^{k+j}}{2}\cot(\frac{(k-j)\pi}{N}), & \text{if } \quad k \neq j \\ 0, & \text{if } \quad k = j \end{cases}$$

Solution:

(a) i. $\hat{f}_k = \frac{1}{N} \sum\limits_{j=0}^{N-1} f(x_j) e^{-ikx_j} = \frac{1}{N} \sum\limits_{j=0}^{N-1} f(x_j) e^{-i(k+N)x_j} = \tilde{f}_{k+N}$ for $k = -\frac{N}{2}+1, ..., -1$

ii. $a_{\pm\frac{N}{2}} = 2$

iii.

$$\sum_{k=0}^{N-1} \tilde{f}_k \bar{\tilde{f}}_k = \frac{1}{N^2} \sum_{j,l=0}^{N-1} f(x_j) \bar{f}(x_l) \sum_{k=0}^{N-1} e^{-i2\pi k \frac{j-l}{N}}$$
$$= \frac{1}{N} \sum_{j,l=0}^{N-1} f(x_j) \bar{f}(x_l) \delta_{jl} = \frac{1}{N} \sum_{k=0}^{N-1} |f_k|^2$$

(b)

$$(I_N(f))(x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{f}^k e^{ikx}$$
$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \left(\frac{1}{Na_k} \sum_{j=0}^{N-1} f(x_j) e^{-ikx_j} \right) e^{ikx}$$
$$= \sum_{j=0}^{N-1} \left(\frac{1}{N} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{a_k} e^{ik(x-x_j)} \right) f(x_j)$$

Therefore,

$$g_j(x) = \frac{1}{N} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{a_k} e^{ik(x-x_j)}$$
$$= \frac{1}{N} \left(\sum_{k=0}^{\frac{N}{2}-1} e^{ik(x-x_j)} + \sum_{k=-\frac{N}{2}+1}^{-1} e^{ik(x-x_j)} + \cos(N\frac{x-x_j}{2}) \right)$$
$$= \frac{1}{N} \left(\frac{1-e^{i\frac{N}{2}(x-x_j)}}{1-e^{i(x-x_j)}} + \frac{e^{-i(x-x_j)} - e^{-i\frac{N}{2}(x-x_j)}}{1-e^{-i(x-x_j)}} + \cos(N\frac{x-x_j}{2}) \right)$$
$$= \frac{1}{N} \left(\frac{\sin((N-1)\frac{x-x_j}{2})}{\sin(\frac{x-x_j}{2})} + \cos(N\frac{x-x_j}{2}) \right)$$
$$= \frac{1}{N} \sin(N\frac{x-x_j}{2}) \cot(\frac{x-x_j}{2})$$

(c) $(I_N(f))^{(m)}(x) = \sum_{n=0}^{N-1} f(x_j) g_j^{(m)}(x)$, so $(I_N(f))^{(m)}(x_k) = \sum_{n=0}^{N-1} f(x_j) g_j^{(m)}(x_k)$ for $k = 0, ..., N-1$
Hence $D^m(k,j) = g_j^{(m)}(x_k)$. In particular, let $\theta = \frac{x-x_j}{2}$, then

$$g`_j(x) = \frac{1}{2}\cos(N\theta)\cot(\theta) - \frac{1}{2N}\sin(N\theta)\csc^2(\theta)$$

if $x = x_k \neq x_j$, then $g_j'(x_k) = \frac{1}{N}\cos(\pi(k-j))\cot(\frac{(k-j)\pi}{N})$; if $k = j$, then

$$\lim_{\theta\to 0} \frac{1}{2}\cos(N\theta)\cot(\theta) - \frac{1}{2N}\sin(N\theta)\csc^2(\theta) = \lim_{\theta\to 0} \frac{1}{2N} \frac{\frac{N}{2}\cos(N\theta)\sin(2\theta) - \sin(N\theta)}{\sin^2(\theta)}$$
$$= \lim_{\theta\to 0} \frac{1}{2N} \frac{-\frac{N^2}{2}\sin(N\theta)\sin(2\theta) + N\cos(N\theta)\cos(2\theta) - N\cos(N\theta)}{\sin(2\theta)}$$
$$= \lim_{\theta\to 0} \frac{1}{2} \frac{\cos(N\theta)\cos(2\theta) - \cos(N\theta)}{\sin(2\theta)}$$
$$= \lim_{\theta\to 0} \frac{1}{2}\cos(N\theta) \frac{-2\sin^2(\theta)}{2\sin(\theta)\cos(\theta)} = 0$$

5. Consider the differential equation:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume u and f are periodically extended to $\mathbb{R}$. Divide the interval $[0, 2\pi]$ into n equal portions and let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$.

for $j = 0, 1, 2, ..., n-1$.

(a) Use $u(x_{j\pm 2})$ to approximate $u'(x_j)$ and use $u(x_{j\pm 4})$ and $u(x_j)$ to approximate $u''(x_j)$ and explain why the corresponding matrices $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Prove that $\overrightarrow{e^{ikx}} := (e^{ikx_0}, e^{ikx_1}, ..., e^{ikx_{n-1}})^T$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for $k = 0, 1, 2, ..., n-1$. What are their corresponding eigenvalues? Please explain your answer with details.

(c) Show that $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ forms a basis for $\mathbb{C}^n$.

(d) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in \mathbb{C}$. If $\mathbf{u}$ satisfies $a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}$, show that

$$(a\lambda_k^2 + b\lambda_k)\hat{u}_k = \hat{f}_k \text{ where } \lambda_k = i\frac{\sin(2kh)}{2h},$$

for $k = 0, 1, 2, ..., n-1$. Please explain your answer with details.

Solution:

(a) By Taylor's expansion, we get

$$u(x_{j+2}) = u(x_j) + 2hu'(x_j) + o(2h)$$
$$u(x_{j-2}) = u(x_j) - 2hu'(x_j) + o(2h)$$

so we deduce that

$$u'(x_j) = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} + o(1)$$

Similarly,

$$u(x_{j+4}) = u(x_j) + 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$
$$u(x_{j-4}) = u(x_j) - 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$

so

$$u''(x_j) = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2} + o(1)$$

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n \times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} \quad \text{and} \quad (\mathcal{D}_2\mathbf{u})_j = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2}.$$

for $j = 0, 1, 2, ..., n-1$.

Then we can say that when we choose n is sufficiently large (or h is sufficiently small), $\mathcal{D}_2\mathbf{u}$ and $\mathcal{D}_1\mathbf{u}$ can approximate $\mathbf{u}''$ and $\mathbf{u}'$, respectively.

(b) By the structure of $\mathcal{D}_1\mathbf{u}$, it can be verified that

$$(\mathcal{D}_1\overrightarrow{e^{ikx}})_j = \frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4h}$$

So it suffices to show that

$$\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}}$$

is independent of the index j, and this value is exactly the eigenvalue of $\mathcal{D}_1$ corresponding $\overrightarrow{e^{ikx}}$.

$$\begin{aligned}\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}} &= \frac{e^{ik\cdot(x_j+2h)} - e^{ik\cdot(x_j-2h)}}{4he^{ikx_j}} \\ &= \frac{e^{i\cdot 2kh} - e^{i\cdot(-2kh)}}{4h} \\ &= \frac{i\sin(2kh)}{2h}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_1$ corresponding the eigenvalue $\frac{i\sin(2kh)}{2h}$ for $k = 0, 1, ..., n-1$. Similarly,

$$\begin{aligned}\frac{e^{ikx_{j+4}} - 2e^{ikx_j} + e^{ikx_{j-4}}}{16h^2e^{ikx_j}} &= \frac{e^{ik\cdot(x_j+4h)} - 2e^{ik\cdot x_j} + e^{ik\cdot(x_j-4h)}}{16h^2e^{ikx_j}} \\ &= \frac{e^{i\cdot 4kh} - 2 + e^{i\cdot(-4kh)}}{16h^2} \\ &= \frac{\cos(4kh) - 1}{8h^2}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_2$ corresponding the eigenvalue $(\frac{i\sin(2kh)}{2h})^2 = \frac{\cos(4kh)-1}{8h^2}$ for $k = 0, 1, ..., n-1$.

(c) Since $\overrightarrow{e^{ikx}}$ are the eigenvectors of $\mathcal{D}_1$ corresponding the distinct eigenvalues, we get that they are linearly independent. So the set contains n linearly independent vectors forms a basis.

(d) By (b) we get $\mathcal{D}_1\overrightarrow{e^{ikx}} = \lambda_k\overrightarrow{e^{ikx}}$, $\mathcal{D}_2\overrightarrow{e^{ikx}} = (\lambda_k)^2\overrightarrow{e^{ikx}}$
so

$$\begin{aligned}a\mathcal{D}_2\mathbf{u} + \mathbf{b}\mathcal{D}_\mathbf{1}\mathbf{u} &= a\mathcal{D}_2(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) + b\mathcal{D}_1(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) \\ &= a\sum_{k=0}^{n-1}(\lambda_k)^2\hat{u}_k\overrightarrow{e^{ikx}} + b\sum_{k=0}^{n-1}\lambda_k\hat{u}_k\overrightarrow{e^{ikx}} \\ &= \sum_{k=0}^{n-1}(a(\lambda_k)^2 + b\lambda_k)\hat{u}_k\overrightarrow{e^{ikx}} \\ &= \mathbf{f} \\ &= \sum_{k=0}^{n-1}\hat{f}_k\overrightarrow{e^{ikx}}\end{aligned}$$

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ is a basis, comparing the coefficients leads to the result that we want to prove.