Lecture 9: Recall:

Main idea of numerical spectral method

Diff. eqt $\longrightarrow$ Algebraic eqt $\xrightarrow[\text{on } \hat{u}]{\text{inverse DFT}}$ solution $\vec{u}$

(e.g. $\frac{d^2u}{dx^2} = f$)

DFT on $\vec{f}$ $\left(\frac{\overline{A_w}}{n}\vec{f}\right)$ Fast?

Fast!

$(A_w \hat{u})$ Fast?

Remark: • To develop an efficient numerical spectral method, we need to compute $A_w \hat{u}$ and $\frac{\overline{A_w}}{n}\vec{f}$ fast.

• Computational cost for $\underset{(n\times n)}{A_w}\hat{u}$ is $O(n^2)$.

Goal: Reduce the computational cost to $O(n \log n)$

e.g. $n = 2^{10}$, $n^2 = 2^{20}$, $n \log n = 10 \cdot 2^{10} < 2^{14}$. $\therefore$ $2^6 = 64$ times faster!

Fast Fourier Transform (FFT) (Colley and Tukey, 1965)

Let $F_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 \cdots & \omega_n^{n-1} \\ 1 & \vdots & & \\ \vdots & \vdots & & \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} \cdots & \omega_n^{(n-1)^2} \end{pmatrix} \in M_{n\times n}$ where $\omega_n = e^{i\left(\frac{2\pi}{n}\right)}$

Let $\vec{y} = F_n \vec{x}$, where $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$. Suppose $n = 2m$.

Then: for each $0 \le j \le n-1$,

$$y_j = \sum_{k=0}^{n-1} \omega_n^{jk} x_k = \sum_{k=0}^{2m-1} \omega_{2m}^{kj} x_k$$

$$\begin{cases} \hat{f}(l) = \sum_{k=0}^{n-1} \hat{f}(k) e^{i\frac{2\pi}{n}kl} \\ \hat{f}(k) = \frac{1}{n}\sum_{l=0}^{n-1} f(l) e^{i\frac{2\pi}{n}kl} \end{cases}$$

Divide $k = 0, 1, 2, \ldots, 2m-1$ into two parts:

Part 1: $0, 2, 4, 6, \ldots, 2(m-1)$ (Even)

Part 2: $1, 3, 5, 7, \ldots, 2m-1$ (Odd)

$$y_j = \underbrace{\sum_{k=0}^{m-1} \omega_{\cancel{n}\;2m}^{2kj}\, x_{2k}}_{\text{Part 1}} + \underbrace{\sum_{k=0}^{m-1} \omega_n^{(2k+1)j}\, x_{2k+1}}_{\text{Part 2}}$$

$$= \sum_{k=0}^{m-1} \omega_m^{kj}\, \overset{(\vec{x}')_k}{x_{2k}} + \sum_{k=0}^{m-1} \omega_n^{j}\, \omega_m^{kj}\, \overset{(\vec{x}'')_k}{x_{2k+1}}$$

$$\left(\because \; \omega_{2m}^{2k} = e^{i\left(\frac{2\pi}{\cancel{2}m}\right)\cancel{2}k} = \omega_m^{k} \right)$$

Denote $\vec{x}' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2m-2} \end{pmatrix}$, $\vec{x}'' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2m-1} \end{pmatrix}$. Denote $\vec{y}' = \underset{M_{m\times m}}{F_m}\,\vec{x}'$ and $\vec{y}'' = F_m\,\vec{x}''$.

$$\therefore \quad y_j = (\overset{\frac{n}{2}\times\frac{n}{2}}{F_m}\,\vec{x}')_j + \omega_n^{j}\,(\overset{\frac{n}{2}\times\frac{n}{2}}{F_m}\,\vec{x}'')_j \quad \text{for } j = 0, 1, 2, \ldots, m-1$$

$$= (\vec{y}')_j + \omega_n^{j}\,(\vec{y}'')_j \qquad j = m, m+1, \ldots, 2m-1$$

$$y_{j+m} = \sum_{k=0}^{m-1} \omega_n^{2k(j+m)} x_{2k} + \sum_{k=0}^{m-1} \omega_n^{(2k+1)(j+m)} x_{2k+1} \quad \text{for } j = 0, 1, 2, \dots, m-1$$

Capture

$y_m, y_{m+1}, \dots, y_{2m-1}$

$$= \sum_{k=0}^{m-1} \omega_m^{kj} \, \underbrace{\omega_{\cancel{2}m}^{\cancel{2}km}}_{= e^{i\left(\frac{2\pi}{\cancel{m}}\right) k\cancel{m}} = 1} (\vec{x}')_k + \sum_{k=0}^{m-1} \omega_m^{k(j+m)} \omega_n^{j+m} (\vec{x}'')_k$$

$$\omega_m^{k(j+m)} = \omega_m^{kj} \, \cancel{\omega_m^{km}} \qquad \omega_m^{km} = 1$$

$$\omega_n^{j+m} = \omega_n^{j} \, \omega_n^{m} \qquad \omega_n^{m} = e^{i\left(\frac{2\pi}{2\cancel{m}}\right)\cdot \cancel{m}}$$

$$\therefore \quad y_{j+m} = \sum_{k=0}^{m-1} \omega_m^{kj} (\vec{x}')_k + \sum_{k=0}^{m-1} (-\omega_n^{j}) \, \omega_m^{kj} (\vec{x}'')_k \qquad \omega_n^m = -1$$

$$y_{j+m} = (F_m \vec{x}')_j - \omega_n^{j} (F_m \vec{x}'')_j \quad \text{for } j = 0, 1, 2, \dots, m-1$$

Note: $n \times n$ matrix multiplication becomes $\frac{n}{2} \times \frac{n}{2} = m \times m$ matrix multiplication.

Denote:
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \\ \vdots \\ v_n w_n \end{pmatrix}$$

Then:
$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} = \underset{F_m \vec{x}'}{\underset{\shortparallel}{\vec{y}'}} + \begin{pmatrix} w_n^0 \\ w_n^1 \\ w_n^2 \\ \vdots \\ w_n^{m-1} \end{pmatrix} \otimes \underset{F_m \vec{x}''}{\underset{\shortparallel}{\vec{y}''}} \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}' - \begin{pmatrix} w_n^0 \\ w_n^1 \\ w_n^2 \\ \vdots \\ w_n^{m-1} \end{pmatrix} \otimes \vec{y}''$$

Summary of FFT $(n = 2m)$

Step 1: Split $\vec{x}$ into $\vec{x}' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2m-2} \end{pmatrix}$, $\vec{x}'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$

Step 2: Compute $\vec{y}' = F_m \vec{x}'$ and $\vec{y}'' = F_m \vec{x}''$

$F_m \rightsquigarrow F_{\frac{m}{2}} \rightsquigarrow F_{\frac{m}{4}} \cdots$

Step 3: Compute
$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} = \underset{F_m \vec{x}'}{\underset{\shortparallel}{\vec{y}'}} + \begin{pmatrix} w_n^0 \\ w_n^1 \\ w_n^2 \\ \vdots \\ w_n^{m-1} \end{pmatrix} \otimes \underset{F_m \vec{x}''}{\underset{\shortparallel}{\vec{y}''}} \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}' - \begin{pmatrix} w_n^0 \\ w_n^1 \\ w_n^2 \\ \vdots \\ w_n^{m-1} \end{pmatrix} \otimes \vec{y}''$$

Remark: Comp. cost : $\mathcal{O}(m^2) + \mathcal{O}(m)$ $\left(\begin{array}{c} m \text{ multiplication} \\ + \\ m \text{ addition} \\ + \\ m \text{ subtraction} \end{array} \right)$

$$\mathcal{O}(m^2) + \mathcal{O}(m) \approx \mathcal{O}(m^2)$$

Remark: If n is even,

$$\underset{}{F_n} \rightsquigarrow \underset{\text{(even)}}{F_{\frac{n}{2}}} \rightsquigarrow \underset{\text{(even)}}{F_{\frac{n}{4}}} \rightsquigarrow F_{\frac{n}{8}} \rightsquigarrow \cdots \rightsquigarrow \underset{\text{scalar}}{F_1}$$

If we choose $n = 2^l$, then the above procedure can be carried out.

Next time: with the above procedure, comp cost $\approx \mathcal{O}(n \log n)$