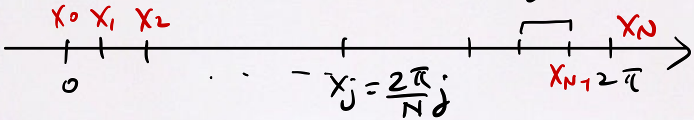
Lecture 7 Recall:

DFT and numerical diff eqt

Consider: $\dfrac{d^2u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition $u(0) = u(2\pi)$

Suppose f is measured only at N discrete points:

$$x_0,\ x_1,\ \dots,\ x_{N-1} \qquad h = \frac{2\pi}{N}$$



Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

(unknown)

Recall:

By Taylor's expansion,

$$u(x_j+h) \approx u(x_j) + h\,u'(x_j) + \frac{h^2}{2}u''(x_j) \qquad \text{——— (1)}$$

$$u(x_j-h) \approx u(x_j) - h\,u'(x_j) + \frac{h^2}{2}u''(x_j) \qquad \text{——— (2)}$$

$$(1)+(2): \quad u''(x_j) \approx \frac{u(\underbrace{x_j-h}_{x_{j-1}}) - 2u(x_j) + u(\overbrace{x_j+h}^{x_{j+1}})}{h^2}$$

$$\therefore \quad u''(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \quad \text{(Central difference approximation)}$$

for $j = 0, 1, 2, \ldots, N-1$

Thus:

$$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{h^2}\begin{array}{c} \begin{array}{ccccc} u_0 & u_1 & \cdots & & u_{N-1} \end{array} \\ \begin{pmatrix} -2 & 1 & \cdots & & \boxed{1} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ \boxed{1} & & & 1 & -2 \end{pmatrix} \end{array}$$

(Use the fact that $u_0 = u_N$, $u_{-1} = u_{N-1}$)

$\therefore \quad \frac{d^2u}{dx^2} = f$ can be discretized as $\boxed{\underset{M_{N\times N}}{\ni}\tilde{D}\vec{u} = \vec{f}}$ (Linear System)

Numerical differential eqt

Remark: $\tilde{D}$ is BIG matrix !!

Goal: Design numerical spectral method to solve $\tilde{D}\vec{u} = \vec{f}$.

Need to: Determine eigenvalues / eigenvectors of $\tilde{D}$

In continuous case, e^{ikx} is an eigenfunction of $\frac{d^2}{dx^2}$, that is periodic.

In discrete case, define:

$$\underset{\substack{\cap \\ \mathbb{C}^N}}{\overrightarrow{e^{ikx}}} \overset{\text{def}}{=\!=} \begin{pmatrix} e^{ikx_0} \\ e^{ikx_1} \\ \vdots \\ e^{ikx_{N-1}} \end{pmatrix}$$

(Capture the values of e^{ikx} at N discrete points)

Claim: $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$ $(k = 0, 1, 2, \ldots, N-1)$

Mr precisely,

$$\tilde{D}\,\overrightarrow{e^{ikx}} = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{= -\lambda_k^2}\overrightarrow{e^{ikx}}$$

where: $\lambda_k^2 = \left(\frac{4\sin^2\frac{kh}{2}}{h^2}\right).$

or $\lambda_k = \left(\frac{2\sin\frac{kh}{2}}{h}\right)$

Claim: $\left\{ \overrightarrow{e^{ikx}} \right\}_{k=0}^{N-1}$ is a basis of $\mathbb{C}^N$ (consisting of eigenvectors)

Pf:

$$\begin{pmatrix} | & | & & | \\ \overrightarrow{e^{i(0)x}} & \overrightarrow{e^{i(1)x}} & \cdots & \overrightarrow{e^{i(N-1)x}} \\ | & | & & | \end{pmatrix} = \underset{\sim}{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix}} = A_\omega \ ; \ \omega = e^{i\frac{2\pi}{N}}$$

$$A_\omega \overline{A_\omega} = N I_{N\times N}$$

$$\Rightarrow \quad A_\omega^{-1} = \frac{1}{N} \overline{A_\omega}$$

Claim: $\text{Rank}(\tilde{D}) = N-1$ and nullspace of $\tilde{D}$ is:

$$N(\tilde{D}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

Proof: $\text{Rank}(\tilde{D}) = \#$ of non-zero eigenvalues of $\tilde{D}$.

$$= \#\{-\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_{N-1}^2\}$$

$$= N-1$$

$$N(\tilde{D}) = \text{eigenspace of eigenvalue} = 0 \quad (\tilde{D}\vec{x} = \vec{0} = 0\vec{x})$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Numerical Spectral method

Since $\{ \overrightarrow{e^{ikx}} \}_{k=0}^{N-1}$ is a basis. We can write:

$$\vec{u} = \sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \quad \text{and} \quad \vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$\in \mathbb{C}^N$ $\quad$ $\in \mathbb{C}$ $\quad$ $\in \mathbb{C}^N$

In other words, for each j, $f_j = f(x_j) = \sum_{k=0}^{N-1} \hat{f}_k \left(\overrightarrow{e^{ikx}}\right)_j$

$$= \sum_{k=0}^{N-1} \hat{f}_k e^{ikx_j} \qquad x_j = \frac{2\pi j}{n}$$

(DFT!)

$\therefore$ $\hat{f}_k$ can be determined by DFT.

To solve $\frac{d^2u}{dx^2} = f$, we approximate it by

$$\tilde{D}\vec{u} = \vec{f}.$$

Now, $\tilde{D}\vec{u} = \vec{f}$ becomes:

$$\tilde{D}\left(\sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}}\right) = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k \underbrace{\tilde{D}\overrightarrow{e^{ikx}}}_{= (-\lambda_k^2)\overrightarrow{e^{ikx}}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \hat{u}_k (-\lambda_k^2) \overrightarrow{e^{ikx}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

Comparing coefficients, we get

$$\underbrace{-\lambda_k^2}_{\text{known}} \underbrace{\hat{u}_k}_{\text{unknown}} = \underbrace{\hat{f}_k}_{\text{known}} \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

(algebraic equation)

$$(-\lambda_k^2)\,\hat{u}_k = \hat{f}_k \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

For $k = 1, 2, \ldots, N-1$, we have: $\hat{u}_k = \hat{f}_k \big/ (-\lambda_k^2)$

For $k = 0$, $\lambda_k = 0$.

We consider a special solution such that:

$$\hat{u}_0 = \frac{u_0 + u_1 + \cdots + u_{N-1}}{N} = 0$$

$$\hat{f}_0 \;//$$

$\therefore$ Set $\hat{u}_0 = 0$

Note that $\hat{f}_0 = -\lambda_0^2\, \hat{u}_0 = 0 \Rightarrow \dfrac{f_0 + f_1 + \cdots + f_{N-1}}{N} = 0$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} u''(x)\,dx = u'(x)\Big|_0^{2\pi} = 0 \quad \text{(periodic)}$$

$$\underset{\hat{f}_0}{\wr\wr}$$

Once $\hat{u}_k$ are all defined for $k=0,1,2,\dots,N-1$.

$\vec{u}$ can be obtained:

$$\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ \vdots \\ u(x_{N-1}) \end{pmatrix} = \overrightarrow{\sum_{k=0}^{N-1} \hat{u}_k e^{ikx}} \quad (\text{inverse DFT})$$

$$\Leftrightarrow \quad \vec{u} = A_w \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \vdots \\ \hat{u}_k \end{pmatrix} \quad \text{where} \quad A_w = \begin{pmatrix} 1 & - & - & - & 1 \\ 1 & w & - & - & w^{N-1} \\ \vdots & & & & \\ 1 & w^{N-1} & - & - & w^{(N-1)^2} \end{pmatrix}$$

Claim: If $\vec{u}_1$ and $\vec{u}_2$ are both solutions of $\tilde{D}\vec{u} = \vec{f}$,

then: $\vec{u}_1 = \vec{u}_2 + c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ for some const. c.

Proof:

$$\begin{array}{rl} & \tilde{D}\vec{u}_1 = \vec{f} \\ - & \tilde{D}\vec{u}_2 = \vec{f} \\ \hline & \tilde{D}(\vec{u}_1 - \vec{u}_2) = \vec{0} \end{array} \implies \vec{u}_1 - \vec{u}_2 \in N(\tilde{D})$$

$\therefore\ \vec{u}_1 = \vec{u}_2 + c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ for some const. c

Remark: For any other sol $\vec{u}^*$ $(\tilde{D}\vec{u}^* = \vec{f})$,

$$\vec{u}^* = \underset{\text{(Special sol)}}{\vec{u}} + c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(c) determined by the particular condition (boundary condition)

Example: Consider: $a\dfrac{d^2u}{dx^2} + b\dfrac{du}{dx} = f$ for $x \in [0, 2\pi]$.

This time, we approximate $\dfrac{d^2u}{dx^2}$ by:

$$(*) \quad \frac{d^2u}{dx^2}(x_j) \approx \frac{u_{j-2} - 2u_j + u_{j+2}}{4h^2} \quad \text{for } j = 0, 1, 2, \ldots, N-1$$

Again, we assume $u_{-1} = u_{N-1}$, $u_1 = u_{N+1}$, $u_{-2} = u_{N-2}$, ..., etc

Motivation: ① $u(x_j + 2h) \approx u(x_j) + 2h\,u'(x_j) + 2h^2 u''(x_j)$

② $u(x_j - 2h) \approx u(x_j) - 2h\,u'(x_j) + 2h^2 u''(x_j)$

① + ② : $u(x_{j+2}) + u(x_{j-2}) - 2u(x_j) = 4h^2 u''(x_j)$

This time, we approximate $\dfrac{du}{dx}$ as:

$$(**) \quad \frac{du}{dx}(x_j) \approx \frac{u_{j+1} - u_{j-1}}{2h}$$

Denote (**) in matrix form as:

$$\begin{pmatrix} u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{N-1}) \end{pmatrix} = \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{2h}\begin{pmatrix} 0 & 1 & & & -1 \\ -1 & 0 & \ddots & O & \\ & \ddots & \ddots & \ddots & \\ & O & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{pmatrix}$$

Denote (*) in matrix form as:

$$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} = D\vec{u} \quad \text{where} \quad D = \frac{1}{4h^2}\begin{pmatrix} -2 & 0 & 1 & & \cdots & 1 & 0 \\ 0 & -2 & 0 & 1 & & & \\ 1 & 0 & \ddots & 0 & \ddots & O & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1 \\ & O & & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & & & 1 & 0 & -2 \end{pmatrix}$$

<u>Remark:</u> $D = \tilde{D}^2$.

<u>Claim:</u> $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$ and D.

<u>Proof:</u>

$$\left(D\overrightarrow{e^{ikx}}\right)_j = \frac{e^{ikx_{j-2}} - 2e^{ikx_j} + e^{ikx_{j+2}}}{4h^2}$$

$$= \frac{e^{ikx_j}\left(e^{-2ikh} - 2 + e^{2ikh}\right)}{4h^2}$$

$$= \underbrace{\left(-\frac{\sin^2(kh)}{h^2}\right)}_{\tilde{\lambda}_k^2} e^{ikx_j}$$

$$\therefore D\overrightarrow{e^{ikx}} = \tilde{\lambda}_k^2 \overrightarrow{e^{ikx}}.$$

Also,

$$\left(\tilde{D}\overrightarrow{e^{ikx}}\right)_j = \frac{e^{ikx_{j+1}} - e^{ikx_{j-1}}}{2h} = e^{ikx_j}\frac{\left(e^{ikh} - e^{-ikh}\right)}{2h}$$

$$= \underbrace{\left(\frac{i\sin(kh)}{h}\right)}_{\tilde{\lambda}_k} e^{ikx_j}$$

$$\therefore \tilde{D}\overrightarrow{e^{ikx}} = \tilde{\lambda}_k \overrightarrow{e^{ikx}}$$