Lecture 6:

Recall:

Definition: (Discrete Fourier Transform) Given $f_0, f_1, \dots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

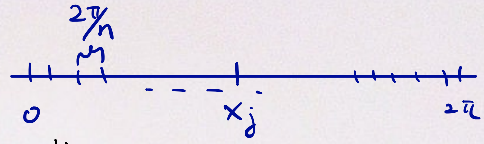
$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)}$$

for $k = 0, 1, 2, \dots, n-1$

The inverse discrete Fourier Transform recovers the original signal:

$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \dots, n-1$$

Motivation 1: Let $f(x)$ defined on $[0, 2\pi]$



Approximate $f(x)$ by:

$$F_n(x) = \sum_{k=0}^{n-1} c_k e^{ikx}, \quad x \in [0, 2\pi] \text{ such that:}$$

$$x_j = j\left(\frac{2\pi}{n}\right)$$

$$\boxed{F_n(x_j) = f(x_j) \overset{\text{def}}{:=} f_j}, \quad x_j = \frac{2\pi}{n}. \quad (\text{for all } j = 0, 1, 2, \ldots, n-1)$$

$$(*)\begin{cases} F_n(\overset{\overset{0}{\|}}{x_0}) = c_0 + c_1 + c_2 + \ldots + c_{n-1} = f_0 \\ F_n(x_1) = c_0 + c_1 e^{ix_1} + c_2 e^{i2x_1} + \ldots + c_{n-1} e^{i(n-1)x_1} = f_1 \\ \vdots \\ F_n(x_{n-1}) = c_0 + c_1 e^{ix_{n-1}} + c_2 e^{i2x_{n-1}} + \ldots + c_{n-1} e^{i(n-1)x_{n-1}} = f_{n-1} \end{cases}$$

Remark: Computational cost for DFT is:

$$\begin{array}{c} n^2 \text{ multiplication} \\ + \\ n(n-1) \text{ addition} \end{array} = \mathcal{O}(n^2)$$

$$A_w = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ & & \vdots & \end{pmatrix}$$

$$\omega = e^{i\frac{2\pi}{4}}$$

Example: Consider $f(t) = 5 + 2\cos(t - \frac{\pi}{2}) + 3\cos(2t)$.

f is 2π-periodic. Divide $[0, 2\pi]$ by 4 partitions. Find the DFT of f (discretized by 4 points).

$f_0 = f(0) = 8$; $f_1 = f\left(\frac{2\pi}{4}\right) = 4$; $f_2 = f\left(\frac{4\pi}{4}\right) = 8$; $f_3 = f\left(\frac{6\pi}{4}\right) = 0$

$\therefore$ DFT : $c_k = \frac{1}{4} \sum_{j=0}^{3} f_j e^{-i\left(\frac{2jk\pi}{4}\right)}$ for $k = 0, 1, 2, 3$ or

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{\overline{A_w}}{\underset{\overset{\shortparallel}{4}}{n}} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}}{4} \begin{pmatrix} 8 \\ 4 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -i \\ 3 \\ i \end{pmatrix}$$

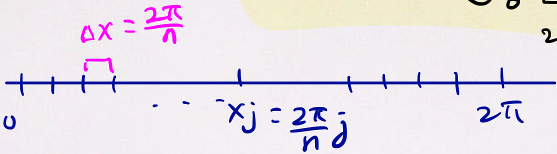
$$\omega = e^{\frac{2\pi i}{4}}$$

Motivation 2: Fourier Transform $\overset{\text{related}}{\longleftrightarrow}$ Fourier Series extended to $(-\infty, \infty)$

Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(x)}_{2\pi\text{-periodic}} e^{-ikx}\, dx$$

Divide:



We can approximate the integration:

$$c_k \approx \frac{1}{2\pi}\sum_{j=0}^{n-1} f(x_j)\, e^{-ikx_j}\,\Delta x = \frac{1}{\cancel{2\pi}}\sum_{j=0}^{n-1} f_j\, e^{-i\frac{2\pi}{n}jk}\left(\frac{\cancel{2\pi}}{n}\right)$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} f_j\, e^{-i\frac{2jk\pi}{n}} \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

$$= \text{DFT}$$

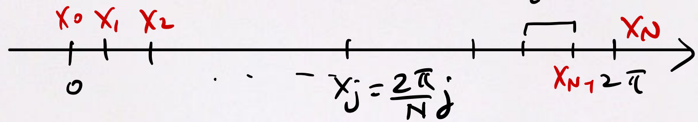
DFT = approximation of (complex) Fourier coefficient.

DFT and numerical diff eqt

Consider: $\dfrac{d^2u}{dx^2} = f$ for $x \in [0, 2\pi]$ with periodic boundary condition $u(0) = u(2\pi)$

Suppose f is measured only at N discrete points:

$$x_0,\ x_1,\ \ldots,\ x_{N-1} \qquad h = \frac{2\pi}{N}$$



Let $\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \in \mathbb{R}^N$ and $\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{N-1}) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \in \mathbb{R}^N$

(unknown)

By Taylor's expansion,

$$u(x_j+h) \approx u(x_j) + h\,u'(x_j) + \frac{h^2}{2}u''(x_j) \qquad \text{——— (1)}$$

$$u(x_j-h) \approx u(x_j) - h\,u'(x_j) + \frac{h^2}{2}u''(x_j) \qquad \text{——— (2)}$$

(1) + (2) :
$$u''(x_j) \approx \frac{u(\underbrace{x_j-h}_{x_{j-1}}) - 2u(x_j) + u(\overbrace{x_j+h}^{x_{j+1}})}{h^2}$$

$$\therefore\ u''(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}$$ (Central difference approximation)

for $j = 0, 1, 2, \ldots, N-1$

Thus:
$$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} \approx \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{h^2}\begin{array}{c} \begin{array}{cccccc} u_0 & u_1 & & \cdots & & u_{N-1} \end{array} \\ \begin{pmatrix} -2 & 1 & & \cdots & & \boxed{1} \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ \boxed{1} & & & & 1 & -2 \end{pmatrix} \end{array}$$

(Use the fact that $u_0 = u_N$, $u_{-1} = u_{N-1}$)

$\therefore \quad \dfrac{d^2u}{dx^2} = f$ can be discretized as $\boxed{\underset{M_{N\times N}}{\tilde{D}} \vec{u} = \vec{f}}$ (Linear System)

Numerical differential eqt

Remark: $\tilde{D}$ is BIG matrix !!

Goal: Design numerical spectral method to solve $\tilde{D}\vec{u} = \vec{f}$.

Need to: Determine eigenvalues / eigenvectors of $\tilde{D}$

In continuous case, e^{ikx} is an eigenfunction of $\dfrac{d^2}{dx^2}$, that is periodic.

In discrete case, define:

$$\underset{\mathbb{C}^N}{\overrightarrow{e^{ikx}}} \stackrel{\text{def}}{=} \begin{pmatrix} e^{ikx_0} \\ e^{ikx_1} \\ \vdots \\ e^{ikx_{N-1}} \end{pmatrix}$$

(Capture the values of e^{ikx} at N discrete points)

Claim: $\overrightarrow{e^{ikx}}$ is an eigenvector of $\tilde{D}$ $(k = 0, 1, 2, \ldots, N-1)$

Mr precisely,

$$\tilde{D}\,\overrightarrow{e^{ikx}} = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{\text{"} \, -\lambda_k^2}\overrightarrow{e^{ikx}}$$

where: $\lambda_k^2 = \left(\frac{4\sin^2\frac{kh}{2}}{h^2}\right).$

or $\lambda_k = \left(\frac{2\sin\frac{kh}{2}}{h^2}\right)$

Proof: For each j, $(\tilde{D}\,\overrightarrow{e^{ikx}})_j = \dfrac{e^{ikx_{j-1}} - 2e^{ikx_j} + e^{ikx_{j+1}}}{h^2}$ $\quad x_{j-1} = x_j - h, \quad x_{j+1} = x_j + h$

$$= e^{ikx_j}\left(\frac{e^{-ikh} - 2 + e^{ikh}}{h^2}\right)$$

$$= e^{ikx_j}\left(\frac{\cos kh - \cancel{i\sin kh} - 2 + \cos kh + \cancel{i\sin kh}}{h^2}\right)$$

$1 - 2\sin^2\frac{kh}{2}$

$$= e^{ikx_j}\left(\frac{2\cos kh - 2}{h^2}\right) = \underbrace{\left(-\frac{4\sin^2\frac{kh}{2}}{h^2}\right)}_{-\lambda_k^2} e^{ikx_j}$$

Let $-\lambda_k^2 = \left(-\dfrac{4\sin^2\frac{kh}{2}}{h^2}\right)$. Then: $\tilde{D}\,\overrightarrow{e^{ikx}} = -\lambda_k^2\,\overrightarrow{e^{ikx}}$.