Lecture 5

Recall:

Definition: (Real Fourier Series)

Consider $f(x) \in V = \{$ real-valued 2π-periodic smooth functions $\}$.

Then, the real Fourier Series of $f(x)$ is given by:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$, where $\{a_k\}$ and $\{b_k\}$ are given by:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Complex Fourier Series)

Consider $f(x) \in W = \{\text{complex-valued } 2\pi\text{-periodic smooth functions}\}$

Then, the complex Fourier Series is given by:

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{where } \{C_k\} \text{ is determined by:}$$

$$C_k = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-\overset{\sqrt{-1}}{\overset{\shortparallel}{i}}kx}\, dx}_{} \quad \left(\text{Here, } e^{ikx} = \cos kx + i \sin kx\right)$$

The integration is computed separately for the real part and imaginary part.

Question: How well does it approximate $f(x)$?

Consider: $V_N = \left\{ F(x) = \sum_{k=0}^{N} A_k \cos kx + B_k \sin kx : A_k, B_k \in \mathbb{R} \right\}$

For any 2π-periodic function, define:

$$\|f - F\|^2 := E(A_0, A_1, \dots, A_N, B_1, B_2, \dots, B_N)$$

$$:= \int_0^{2\pi} \left(f(x) - \left(\sum_{k=0}^{N} A_k \cos kx + B_k \sin kx \right) \right)^2 dx$$

Remark: $\|f - F\|$ is called the least square error between f and F.

Theorem: $E(a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N) = \min\limits_{\forall A_k, B_k \in \mathbb{R}} E(A_0, \ldots, A_N, B_1, \ldots, B_N)$

where:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \;; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \;; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Proof: Assume $A_0, \ldots, A_N, B_1, \ldots, B_N$ are the minimizer of E.

Then: $\frac{\partial E}{\partial A_i} = 0 \;; \; \frac{\partial E}{\partial B_i} = 0.$

$$\frac{\partial E}{\partial A_k} = \frac{\partial}{\partial A_k}\int_0^{2\pi}\left(f(x) - \left(\sum_{j=0}^{N} A_j \cos jx + B_j \sin jx\right)\right)^2 dx$$

$$= -2\int_0^{2\pi}\left(f(x) - \sum_{j=0}^{N} A_j \cos jx + B_j \sin jx\right)\cos kx$$

$$= -2\int_0^{2\pi} f(x)\cos kx\,dx + 2\pi A_k = 0 \Rightarrow A_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$$

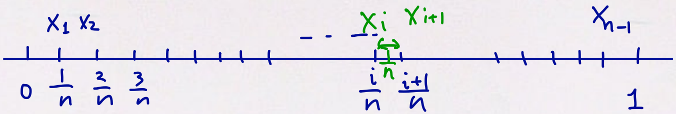
Similarly, $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$ etc ...

Is this the critical point of the minimizer? HW.

Discretization of differential operators

$$\frac{d^2 f}{dx^2}(x) = g(x) \qquad 0 < x < 1 \qquad \text{with} \quad f(0) = 1 \; ; \; f(1) = 2 .$$

Discretize $(0, 1)$:



Approximation of $\frac{d^2 f}{dx^2}$:

$$f(\underbrace{x_{i+1}}_{= x_i + \frac{1}{n}}) \approx f(x_i) + \frac{1}{n} f'(x_i) + \frac{1}{2!}\frac{1}{n^2} f''(x_i) \qquad \text{(Taylor's expansion)}$$

$$+ \; f(\underbrace{x_{i-1}}_{= x_i - \frac{1}{n}}) \approx f(x_i) - \frac{1}{n} f'(x_i) + \frac{1}{2!}\frac{1}{n^2} f''(x_i)$$

$$f(x_{i+1}) + f(x_{i-1}) \approx 2 f(x_i) + \frac{1}{n^2} f''(x_i)$$

$$\therefore \quad \frac{d^2 f}{dx^2}(x_i) \approx \frac{f(x_{i+1}) - 2 f(x_i) + f(x_{i-1})}{\left(\frac{1}{n^2}\right)}$$

$$\therefore \quad \frac{d^2 f}{dx^2}(x_i) = g(x_i) \iff \left\{ \begin{array}{l} \vdots \\ \dfrac{f(x_{i+1}) - 2 f(x_i) + f(x_{i-1})}{\left(\frac{1}{n^2}\right)} = g(x_i) \\ \vdots \end{array} \right. \quad \text{for } i = 1, 2, \ldots, n-1$$

$$D \vec{f} = \vec{g} \; : \; \vec{f} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \end{pmatrix} ; \; \vec{g} = \begin{pmatrix} g(x_1) \\ \vdots \\ g(x_{n-1}) \end{pmatrix}$$

In discrete case, a differential equation can be discretized as:

$$D\vec{u} = \vec{g}$$

where $\vec{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix}$ = values of u at N points $\{x_1, x_2, \dots, x_N\}$

$\vec{g} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}$ = values of g at N points $\{x_1, x_2, \dots, x_N\}$

D = $N \times N$ matrice approximating the differential operator.

<u>Question:</u> Can we "transform" $\vec{u}$ and $\vec{g}$ to turn the (BIG) linear system to SIMPLE algebraic equation?

<u>Answer:</u> YES! Discrete Fourier Transform!!

Question: Extension to discrete case (Computational Math.)

Answer: Discrete Fourier Transform

Goal: ① Define discrete Fourier Transform (DFT)

② Use DFT to solve discretized differential eqt.

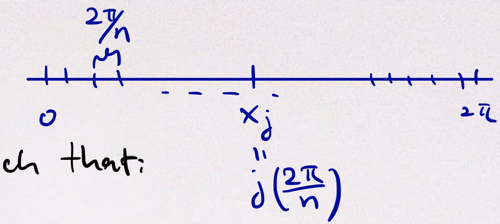
Definition: (Discrete Fourier Transform) Given $f_0, f_1, \ldots, f_{n-1} \in \mathbb{C}$, then the discrete Fourier Transform (DFT) is defined as:

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \in \mathbb{C}^n \quad \text{where} \quad c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

The inverse discrete Fourier Transform recovers the original signal:

$$f_j = \sum_{k=0}^{n-1} c_k \, e^{i\left(\frac{2jk\pi}{n}\right)} \quad \text{for } j = 0, 1, 2, \ldots, n-1$$

Motivation 1: Let $f(x)$ defined on $[0, 2\pi]$



Approximate $f(x)$ by:

$$F_n(x) = \sum_{k=0}^{n-1} c_k e^{ikx}, \quad x \in [0, 2\pi] \text{ such that:}$$

$$\boxed{F_n(x_j) = f(x_j) \overset{\text{def}}{:=} f_j}, \quad x_j = \frac{2\pi}{n}. \quad (\text{for all } j = 0, 1, 2, \ldots, n-1)$$

$$(*) \begin{cases} F_n(\underset{0}{x_0}) = c_0 + c_1 + c_2 + \ldots + c_{n-1} = f_0 \\ F_n(x_1) = c_0 + c_1 e^{ix_1} + c_2 e^{i2x_1} + \ldots + c_{n-1} e^{i(n-1)x_1} = f_1 \\ \vdots \\ F_n(x_{n-1}) = c_0 + c_1 e^{ix_{n-1}} + c_2 e^{i2x_{n-1}} + \ldots + c_{n-1} e^{i(n-1)x_{n-1}} = f_{n-1} \end{cases}$$

Let $\omega = e^{i\frac{2\pi}{n}} = e^{ix_1}$, $\omega^2 = e^{i\frac{4\pi}{n}} = e^{ix_2}$, $\omega^3 = e^{ix_3}$, ... etc.

$\therefore$ (*) can be written as:

$$\boxed{A_\omega = \begin{pmatrix} 1 & 1 & - & - & - & 1 \\ 1 & \omega & \omega^2 & - & - & - & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & - & - & - & \omega^{2(n-1)} \\ & & \vdots & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & & \omega^{(n-1)^2} \end{pmatrix}} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

$$\therefore \left(A_\omega \overline{A_\omega}\right)_{j,k} = 1\cdot 1 + \omega^j \bar{\omega}^k + \omega^{2j}\bar{\omega}^{2k} + \dots + \omega^{(n-1)j}\bar{\omega}^{(n-1)k}$$

$$= 1 + e^{\frac{2\pi i (j-k)}{n}} + e^{\frac{4\pi i (j-k)}{n}} + \dots + e^{\frac{2\pi i (n-1)(j-k)}{n}}$$

$$= \begin{cases} n & \text{if } j = k \\ \dfrac{1 - \left(e^{\frac{2\pi i (j-k)}{\cancel{n}}}\right)^{\cancel{n}}}{1 - e^{2\pi i (j-k)/n}} = 0 & \text{if } j \neq k \end{cases}$$

with $\left(e^{2\pi i (j-k)}\right) \to 1$

$$\therefore \quad A_\omega \overline{A_\omega} = nI = \overline{A_\omega} A_\omega$$

We have:

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = A_\omega^{-1} \begin{pmatrix} f_0 \\ \vdots \\ f_{n-1} \end{pmatrix} = \frac{\overline{A_\omega}}{n} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

$$\therefore \quad c_k = \frac{1}{n}\left(f_0 + e^{-\frac{2\pi i}{n}k} f_1 + \cdots + e^{-\frac{2\pi i}{n}k(n-1)} f_{n-1} \right)$$

for $k = 0, 1, 2, \ldots, n-1$

$$\parallel$$

DFT