

Lecture 13:

Recall:

- Want to solve: $A\vec{x} = \vec{f}$.
Choose N s.t. $A = N - P$.
Splitting method: $N\vec{x}^{k+1} = P\vec{x}^k + \vec{f}$
- Jacobi: $N = D =$ diagonal part of A
Gauss-Seidel: $N = L + D =$ lower part + diagonal part of A .
- A is SDD $\Rightarrow$ $\begin{matrix} \text{Jacobi} \\ + \\ \text{Gauss-Seidel} \end{matrix}$ converge !!

Note: Previous example requires that $M = N^{-1}P$ is diagonalizable.

What if $N^{-1}P$ is NOT diagonalizable?

Theorem: Let $A \in M_{n \times n}(\mathbb{C})$ be a complex-valued matrix.

Then: $\lim_{k \rightarrow \infty} A^k = 0$ if and only if $\rho(A) < 1$.

Simple consequence:

Corollary: The iterative scheme $\vec{x}^{k+1} = M\vec{x}^k + \vec{b}$ converges if and only if $\rho(M) < 1$.

Proof: Consider
$$\begin{array}{rcl} \vec{x}^{k+1} & = & M\vec{x}^k + \vec{b} \\ \vec{x}^* & = & M\vec{x}^* + \vec{b} \end{array}$$

$\vec{x}^* = \text{sol of } A\vec{x} = \vec{f}$.

$$\therefore \vec{e}^{k+1} = M\vec{e}^k \Rightarrow \vec{e}^k = M^k \vec{e}^0$$

$\vec{e}^k \rightarrow \vec{0}$ iff $M^k \rightarrow 0$ iff $\rho(M) < 1$.

Proof of Theorem

($\Rightarrow$) Let λ be an eigenvalue of A with eigenvector $\vec{v}$.

Then: $A^k \vec{v} = \lambda^k \vec{v}$. Thus, $\lim_{k \rightarrow \infty} A^k \vec{v} = \lim_{k \rightarrow \infty} \lambda^k \vec{v}$

$$\begin{aligned} \vec{0} &= \left(\lim_{k \rightarrow \infty} \lambda^k \right) \vec{v} \neq \vec{0} \\ &\quad \text{"0"} \end{aligned}$$

$$\therefore |\lambda| < 1 \quad (\text{for all eigenvalue } \lambda)$$

$$\therefore \rho(A) < 1$$

$$\text{"max}_j \{ |\lambda_j| = \lambda_j \text{ is eigenvalue} \}$$

($\Leftarrow$) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of A (can be repeated)

We apply the idea of Jordan Canonical Form

Find the invertible Q such that $Q^{-1} A Q$ becomes

Simple

ALMOST looks like a diagonal matrix.

Useful Fact: (Can be used without proof)

For any $A \in M_{n \times n}(\mathbb{C})$, there exists an invertible $Q \in M_{n \times n}(\mathbb{C})$

such that $A = Q J Q^{-1}$ where J is the Jordan Canonical Form of A . Actually,

$$J = \begin{pmatrix} J_{m_1}(\lambda_{i_1}) & & & \\ & J_{m_2}(\lambda_{i_2}) & & \\ & & \ddots & \\ & & & J_{m_s}(\lambda_{i_s}) \end{pmatrix}$$

(λ_{i_j} are eigenvalues of A)

where $J_{m_j}(\lambda_{i_j}) = \begin{pmatrix} \lambda_{i_j}' & & 0 \\ & \ddots & \\ 0 & & \lambda_{i_j}' \end{pmatrix} \in M_{m_j \times m_j}(\mathbb{C})$

is called Jordan block with λ_{i_j}

$$1 \leq j \leq s$$

e.g.

$$\overline{J} = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \end{pmatrix}$$

Now, $A^k = Q J^k Q^{-1}$ and

$$J^k = \begin{pmatrix} J_{m_1}^k(\lambda_{i_1}) & & \\ & \ddots & \\ & & J_{m_s}^k(\lambda_{i_s}) \end{pmatrix}$$

For $k \geq m_j - 1$

$$J_{m_j}^k(\lambda_{ij}) = \begin{pmatrix} \lambda_{ij}^k & C_1^k \lambda_{ij}^{k-1} & C_2^k \lambda_{ij}^{k-2} & \cdots & C_{m_j-1}^k \lambda_{ij}^{k-m_j+1} \\ & \lambda_{ij}^k & C_1^k \lambda_{ij}^{k-1} & \cdots & C_{m_j-2}^k \lambda_{ij}^{k-m_j+2} \\ & & \ddots & \ddots & \vdots \\ & & & \lambda_{ij}^k & C_1^k \lambda_{ij}^{k-1} \\ & & & & \lambda_{ij}^k \end{pmatrix} \quad (\text{By M.I.})$$

$$\because \rho(A) < 1 \quad \therefore |\lambda_{ij}| < 1 \text{ for all } j.$$

$$\therefore \lim_{k \rightarrow \infty} J_{m_j}^k(\lambda_{ij}) = 0 \text{ for all } j$$

$$\text{and } \lim_{k \rightarrow \infty} J^k = 0$$

$$\therefore \lim_{k \rightarrow \infty} A^k = \lim_{k \rightarrow \infty} Q J^k Q^{-1} = 0$$

Splitting choice 3: Successive overrelaxation method (SOR)

Consider the iterative scheme =

(Suppose $A = L + D + U$)

$$L \vec{x}^{k+1} + D \vec{y}^{k+1} + U \vec{x}^k = \vec{b} \quad (*)$$

$$\vec{x}^{k+1} = \vec{x}^k + \omega (\vec{y}^{k+1} - \vec{x}^k) \quad (**)$$

$$\Leftrightarrow \vec{y}^{k+1} = \frac{1}{\omega} (\overset{\text{IR}}{\vec{x}}^{k+1} + (\omega - 1) \vec{x}^k)$$

Putting (**) into (*):

$$(L + \frac{1}{\omega} D) \vec{x}^{k+1} + \frac{1}{\omega} (\omega U + (\omega - 1) D) \vec{x}^k = \vec{b}$$

$$\text{or } \underbrace{(L + \frac{1}{\omega} D)}_N \vec{x}^{k+1} = \underbrace{(\frac{1}{\omega} D - (D + U))}_P \vec{x}^k + \vec{b}$$

Remark: • SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \frac{1}{\omega} D\right)} - \underbrace{\left(\frac{1}{\omega} D - (D+u)\right)}$$

$$N \vec{x}^{k+1} = P \vec{x}^k + \vec{b} \Leftrightarrow \vec{x}^{k+1} = N^{-1} P \vec{x}^k + N^{-1} \vec{b}$$

$$\bullet \rho(N^{-1}P) = \rho\left(\underbrace{\left(L + \frac{1}{\omega} D\right)^{-1} \left(\frac{1}{\omega} D - (D+u)\right)}_{\parallel}\right)$$

has to be strictly less than M_{ω}

1 and it has to be small in order to converge fast.

• If $\omega = 1$, SOR = Gauss-Seidel

Remark: SOR is equivalent to:

$$A = N - P = \underbrace{\left(L + \frac{1}{\omega}D\right)}_N - \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_P$$

Or equivalently, $A = (a_{ij})$

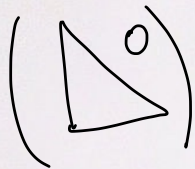
$$\left\{ \begin{array}{l} a_{11} y_1^{k+1} + a_{12} x_2^k + \dots + a_{1n} x_n^k = b_1 \text{ for } x_1^{k+1} = x_1^k + \omega(y_1^{k+1} - x_1^k) \\ a_{21} x_1^{k+1} + a_{22} y_2^{k+1} + a_{23} x_3^k + \dots + a_{2n} x_n^k = b_2 \text{ for } x_2^{k+1} = x_2^k + \omega(y_2^{k+1} - x_2^k) \\ \vdots \\ a_{n1} x_1^{k+1} + a_{n2} x_2^{k+1} + \dots + a_{nn} y_n^{k+1} = b_n \text{ for } x_n^{k+1} = x_n^k + \omega(y_n^{k+1} - x_n^k) \end{array} \right.$$

• SOR = Gauss-Seidel if $\omega = 1$.

Condition for convergence

Theorem: The necessary condition (not sufficient) for SOR to converge is $0 < \omega < 2$.

Proof: Consider: $\det(N^{-1}P) = \text{product of eigenvalues of } N^{-1}P$



$$= \det\left(\left(L + \frac{1}{\omega}D\right)^{-1} \left(\frac{1}{\omega}D - (D+U)\right)\right)$$

$$= \det\left(\left(\frac{1}{\omega}D\right)^{-1}\right) \det\left(\frac{1}{\omega}D - D\right)$$

$$= \det(\cancel{\omega D}) \det\left(\frac{1}{\cancel{\omega}}(1-\omega)\cancel{D}\right)$$

$$\therefore |\det(N^{-1}P)| = \prod_{i=1}^n |\lambda_i| = |1-\omega|^n$$

$\lambda_i = \text{eigenvalue of } N^{-1}P$

$$|1-\omega|^n = \prod_{i=1}^n |\lambda_i| \leq \left(\max_i |\lambda_i|\right)^n = \rho(N^{-1}P)^n$$

$$\rho(N^{-1}P) \geq |1-\omega|$$

Now, SOR converges iff $\rho(N^{-1}P) < 1$

$$\therefore |1-\omega| \leq \rho(N^{-1}P) < 1 \Rightarrow 0 < \omega < 2.$$

Remark: In general, the convergence and convergence rate depends on:

$$\rho(N^{-1}P) < 1$$

Example: Go back: $A\vec{x} = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} := \vec{b}$.

Recall: $\rho(M_J) = \frac{1}{10}$; $\rho(M_{G-s}) = \frac{1}{100}$

$\overset{||}{N_J^{-1}} P_J$ $\overset{||}{N_{G-s}^{-1}} P_{G-s}$

For SOR, $M_{SOR} = (L + \frac{1}{\omega} D)^{-1} (\frac{1}{\omega} D - (D+U))$

$$= \left(\frac{1}{\omega} (D + \omega L) \right)^{-1} \left(\frac{1}{\omega} (D - \omega(D+U)) \right)$$

$$= \cancel{(D + \omega L)^{-1}} \cancel{\left(\frac{1}{\omega} \right)} ((1-\omega)D - \omega U)$$

$$=$$

$\therefore M_{SOR} = \begin{pmatrix} 1-\omega & -\omega/10 \\ -\frac{\omega(1-\omega)}{10} & \frac{\omega^2}{100} + (1-\omega) \end{pmatrix}$

∴ Characteristic polynomial of M_{SOR} :

$$\left[(1-\omega) - \lambda \right] \left[\frac{\omega^2}{100} + 1 - \omega - \lambda \right] - \frac{\omega^2(1-\omega)}{100} = 0 \quad (**)$$

Solving it:

$$\lambda = (1-\omega) + \frac{\omega^2}{200} \pm \frac{\omega}{20} \sqrt{4(1-\omega) + \frac{\omega^2}{100}}$$

Remark: • Adjusting ω gives different eigenvalues and different $\rho(M_{SOR})$

• Put $\omega = 1$ (G-S), $\lambda = 0$, $\lambda = \frac{1}{100}$

• Choose ω such that (**) has a repeated root. That is,

$$4(1-\omega) + \frac{\omega^2}{100} = 0 \Rightarrow \omega = 1.002512579$$

$$\text{Then: } \lambda = (1-\omega) + \frac{\omega^2}{200} = (1-\omega) - 2(1-\omega) = \omega - 1 = 0.002512579$$

$$\rho(M_J) = 0.1, \quad \rho(M_{G-S}) = 0.01, \quad \rho(M_{SOR}) = 0.002512579$$

Remark: For some special matrix, optimal ω can be explicitly found.