Lecture 12

Useful Theorem for eigenvalues

Gerschgorin Theorem

Consider $\vec{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$ = eigenvector of $A = (a_{ij})_{1 \le i, j \le n}$ with eigenvalue λ.

$\in M_{n \times n}(\mathbb{C})$

Then: $A\vec{e} = \lambda \vec{e}$.

For $1 \le i \le n$, $\displaystyle\sum_{j=1}^{n} a_{ij} e_j = \lambda e_i$

$$\Leftrightarrow \quad a_{ii} e_{ii} + \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} e_j = \lambda e_i$$

$$\Leftrightarrow \quad (a_{ii} - \lambda) e_{ii} = -\sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} e_j$$

$$\therefore \quad |a_{ii} - \lambda| \, |e_i| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \, |e_j|$$

Let ℓ be the index such that $|e_\ell| \geq |e_j|$ for $\forall j$

$$|e_\ell| \neq 0$$

Then: $$|a_{\ell\ell} - \lambda|\,\cancel{|e_\ell|} \leq \sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}|\,|e_j| \leq \sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}|\,\cancel{|e_\ell|}$$

$$\therefore \quad |a_{\ell\ell} - \lambda| \leq \sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}| \quad \therefore \quad \lambda \in \overline{B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}|\right)}$$

Ball of radius $\sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}|$ centered at $a_{\ell\ell}$

Note: We don't know ℓ unless we know λ and $\vec{e}$.

BUT: we can conclude:

$$\lambda \in \overline{B_{a_{\ell\ell}}\left(\sum_{\substack{j=1 \\ j\neq \ell}}^{n} |a_{\ell j}|\right)} \subseteq \bigcup_{i=1}^{n} \overline{B_{a_{ii}}\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\right)}$$

Example: Let $A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ 0 & & -1 & 2 \end{pmatrix}$

$$\left(\text{Eigenvalues}: \begin{array}{l} \lambda_1 = 3.618 \\ \lambda_2 = 2.618 \\ \lambda_3 = 1.382 \\ \lambda_4 = 0.382 \end{array} \right)$$

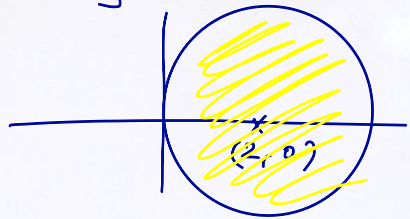
For $l = 1, 4$,

$$\underline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{4} |a_{lj}| \right)} = \{ \lambda : |\lambda - 2| \leq 1 \}$$

For $l = 2, 3$,

$$\underline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \neq l}}^{4} |a_{lj}| \right)} = \{ \lambda : |\lambda - 2| \leq 2 \}$$

$$\therefore \bigcup_{i=1}^{4} \underline{B_{a_{ii}}\left(\sum_{\substack{j=1 \\ j \neq i}}^{4} |a_{ij}| \right)} = \{ \lambda : |\lambda - 2| \leq 2 \}$$



$\because$ A is symmetric, all eigenvalues are real,

$\therefore$ all eigenvalues are between 0 and 4

$\therefore$ $\rho(A) \leq 4$

Condition for Jacobi / Gauss-Seidel to converge

Definition: A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called strictly diagonally dominant (SDD) if $|a_{ii}| > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$ for $i = 1, 2, \dots, n$

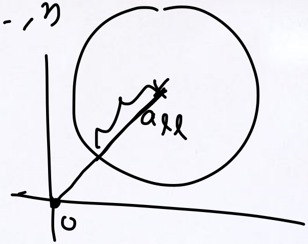
~~(SPD)~~

Theorem: If a matrix A is SDD, then A is non-singular.

Proof: All eigenvalues $\lambda \in \bigcup_{l=1}^{n} \overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \ne l}}^{n} |a_{lj}|\right)}$

A is SDD iff $|a_{ll}| > \sum_{\substack{j=1 \\ j \ne l}}^{n} |a_{lj}|$ for $l = 1, 2, \dots, n$

$\therefore$ Every $\overline{B_{a_{ll}}\left(\sum_{\substack{j=1 \\ j \ne l}}^{n} |a_{lj}|\right)}$ must NOT contain 0.



$\therefore$ No eigenvalue is 0 and so A is non-singular.

Lecture 14:

Theorem: The Jacobi method converges if A is SDD. (Solve: $A\vec{x}=\vec{f}$)

Proof: Note: $\cancel{a_{ii}}\, x_i^{m+1} = -\frac{1}{a_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^m + \frac{f_i}{a_{ii}}$ $(*)$ for $i=1,2,\dots,n$.

Let $\vec{x}^* =$ sol of $A\vec{x}=\vec{f}$, we have:

$$x_i^* = -\frac{1}{a_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^* + \frac{f_i}{a_{ii}} \quad (**) \quad \text{for } i=1,2,\dots,n$$

$(*)-(**)$ $$e_i^{m+1} = -\frac{1}{a_{ii}}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}e_j^m$$

$$\therefore \left|e_i^{m+1}\right| \le \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}\left|e_j^m\right| \le \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}\|\vec{e}^{\,m}\|_\infty \le r\|\vec{e}^{\,m}\|_\infty \quad \text{for all } i$$

where $r = \max_i \left\{ \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right\} < 1$ $\left(\|\vec{e}^{\,m}\|_\infty \overset{\text{def}}{=} \max_j \left\{ |e_j^m| \right\} \right)$

$\therefore \quad \|\vec{e}^{\,m+1}\|_\infty \le \overset{<1}{r}\,\|\vec{e}^{\,m}\|_\infty$

$\therefore \quad \underset{\substack{\downarrow \text{ as } m \to \infty \\ \vec{0}}}{\|\vec{e}^{\,m}\|_\infty} \le r^m \|\vec{e}^{\,0}\|_\infty \longrightarrow 0 \quad \text{as } m \to \infty$

$\therefore$ Jacobi method converges.

<u>Theorem</u>: The Gauss-Seidel method converges to the solution of $A\vec{x} = \vec{f}$ if A is SDD.

<u>Proof</u>: Gauss-Seidel method can be written as:

$$x_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{m} + \frac{f_i}{a_{ii}} \quad \text{---} \ (*)$$

for $i = 1, 2, \dots, n$

Let $\vec{x}^* =$ sol of $A\vec{x} = \vec{f}$. Then:

$$x_i^* = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^* - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^* + \frac{f_i}{a_{ii}} \quad \text{---} \ (**)$$

$(*) - (**)$:

$$e_i^{m+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{m+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{m}$$

We'll prove:

$$\|\vec{e}^{\,m+1}\|_\infty \leq r \|\vec{e}^{\,m}\|_\infty \quad \text{where} \quad r = \max_i \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right\} < 1$$

Induction on i:

When $i=1$, $|e_1^{m+1}| \le \sum_{j=2}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| |e_j^m| \le \|\vec{e}^{\,m}\|_\infty \sum_{j=2}^{n} \left|\frac{a_{ij}}{a_{ii}}\right|$

$\le r\|\vec{e}^{\,m}\|_\infty$

$\therefore$ the statement is true when $i=1$.

Assume $|e_k^{m+1}| \le r\|\vec{e}^{\,m}\|_\infty$ for $k=1,2,\dots,i-1$

Then: $$|e_i^{m+1}| \le \sum_{j=1}^{i-1}\left|\frac{a_{ij}}{a_{ii}}\right| \underbrace{\boxed{|e_j^{m+1}|}}_{\le r\|\vec{e}^{\,m}\|_\infty} + \sum_{j=i+1}^{n}\left|\frac{a_{ij}}{a_{ii}}\right| \underbrace{\boxed{|e_j^{m}|}}_{\le \|\vec{e}^{\,m}\|_\infty}$$

$$\le \underbrace{r\|\vec{e}^{\,m}\|_\infty}_{\le \|\vec{e}^{\,m}\|_\infty} \sum_{j=1}^{i-1}\left|\frac{a_{ij}}{a_{ii}}\right| + \|\vec{e}^{\,m}\|_\infty \sum_{j=i+1}^{n}\left|\frac{a_{ij}}{a_{ii}}\right|$$

$$\le \|\vec{e}^{\,m}\|_\infty \underbrace{\sum_{\substack{j=1\\ j\ne i}}^{n}\left|\frac{a_{ij}}{a_{ii}}\right|}_{\le r} \le r\|\vec{e}^{\,m}\|_\infty$$

By M.I., $|e_i^{m+1}| \leq r \|\vec{e}^{\,m}\|_\infty$ for $i = 1, 2, \ldots, n$

$\therefore \quad \|\vec{e}^{\,m+1}\|_\infty \leq r \|\vec{e}^{\,m}\|_\infty$

$\Rightarrow \quad \|\vec{e}^{\,m}\|_\infty \leq r^m \|\vec{e}^{\,0}\|_\infty \to \vec{0}$ as $m \to \infty$

as $r < 1$.

$\therefore$ G-S. converges!!

Example: Consider $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \vec{b}$

A is SDP. $\therefore$ Both Jacobi and G-S method converges.

For Jacobi method, $\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Let $M_J = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ -1/10 & 0 \end{pmatrix}$

Eigenvalues of M_J are $\lambda_1 = 1/10$ and $\lambda_2 = -1/10$. $\therefore M_J$ is diagonalizable.

$\therefore \rho(M_J) = 1/10$. $\therefore \|\vec{e}^{m}\|_\infty \leq \left(1/10\right)^m K_J$ ← const depending on the initial error $\vec{e}^{0}$

Recall: $\vec{e}^{m} = \lambda_1^m\left(a_1\vec{u}_1 + \sum_{i=2}^{n} a_i\left(\frac{\lambda_i}{\lambda_1}\right)^m \vec{u}_i\right)$ where $\vec{e}^{0} = \sum_{i=1}^{n} a_i \vec{u}_i$

For Gauss-Seidel, $\vec{x}^{k+1} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \vec{x}^{k} + \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Let $M_{G-S} = \begin{pmatrix} 10 & 0 \\ 1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/10 \\ 0 & 1/100 \end{pmatrix}$.

Eigenvalues of M_{G-S} are $\lambda_1 = 1/100$ and $\lambda_2 = 0$ $\therefore$ M_{G-S} is diagonalizable

$\therefore$ $\rho(M_{G-S}) = 1/100$ $\therefore$ $\|\vec{e}^{\,m}\|_\infty \le \left(\frac{1}{100}\right)^m K_{G-S}$ $\leftarrow$ const. depending on initial error.

$\therefore$ G-S converges faster.

<u>Remark</u>: To reduce the error by a factor of 10^{-m}, we need about $k \ge \dfrac{m}{-\log_{10}(\rho(M))}$ iterations.

Jacobi : $k \ge \dfrac{m}{-\log_{10}(1/10)} = m$

G-S : $k \ge \dfrac{m}{-\log_{10}(1/100)} = \dfrac{m}{2}$

$\therefore$ G-S converges twice faster than the Jacobi