Lecture 10: Recall:

Fast Fourier Transform (FFT) (Colley and Tukey, 1965)

Let $F_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \cdots & \omega_n^{n-1} \\ \vdots & \vdots & & \\ 1 & \omega_n^{n-1} & \cdots & \omega_n^{(n-1)^2} \end{pmatrix}$ where $\omega_n = e^{i\left(\frac{2\pi}{n}\right)}$.

Let $\vec{y} = F_n \vec{x}$, where $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$. Suppose $n = 2m$.

Then, for each $0 \leq j \leq n-1$,

$$y_j = \sum_{k=0}^{n-1} \omega_n^{jk} x_k = \sum_{k=0}^{2m-1} \omega_{2m}^{kj} x_k .$$

Divide $k = 0, 1, 2, \ldots, 2m-1$ into two parts:

Part 1: $0, 2, 4, 6, \ldots, 2(m-1)$ (Even)

Part 2: $1, 3, 5, 7, \ldots, 2m-1$ (Odd)

Then: $y_j = \underbrace{\sum_{k=0}^{m-1} \omega_n^{2kj} x_{2k}}_{\text{Part 1}} + \underbrace{\sum_{k=0}^{m-1} \omega_n^{(2k+1)j} x_{2k+1}}_{\text{Part 2}}$

Denote $\vec{x}' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2m-2} \end{pmatrix}$, $\vec{x}'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$. Let $\vec{y}' = F_m \vec{x}'$ and $\vec{y}'' = F_m \vec{x}''$.

j-th entry of $\vec{y}''$

For $j = 0, 1, 2, \ldots, m-1$, we have:

$$y_j = (\overbrace{F_m}^{m \times m} \vec{x}')_j + \omega_n^j (\overbrace{F_m}^{m \times m} \vec{x}'')_j = (\vec{y}')_j + \omega_n^j (\vec{y}'')_j$$

j-th entry of $\vec{y}'$

For $j = 0, 1, 2, \ldots, m-1$, we have

$$y_{j+m} = (\underbrace{F_m}_{m \times m} \vec{x}')_j - \omega_n^j (\underbrace{F_m}_{m \times m} \vec{x}'')_j$$

Note: $n \times n$ matrix multiplication becomes $\frac{n}{2} \times \frac{n}{2} = m \times m$ matrix multiplication.

For simplicity, we denote: $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \\ \vdots \\ v_n w_n \end{pmatrix}$

Summary of FFT

Step 1: Split $\vec{x}$ into $\vec{x}\,' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2(m-1)} \end{pmatrix}$ and $\vec{x}\,'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$

Step 2: Compute $\underset{O(m^2)}{\vec{y}\,' = F_m \vec{x}\,'}$ and $\underset{O(m^2)}{\vec{y}\,'' = F_m \vec{x}\,''}$, where $F_m = \overset{m \times m}{\overset{\parallel}{\frac{n}{2} \times \frac{n}{2}}}$ matrix

Step 3: Compute:

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{pmatrix} = \vec{y}\,' + \overbrace{\begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix}}^{\vec{\omega}_n} \otimes \vec{y}\,'' \quad O(m)$$

and

$$\begin{pmatrix} y_m \\ y_{m+1} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}\,' - \overbrace{\begin{pmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{m-1} \end{pmatrix}}^{\vec{\omega}_n} \otimes \vec{y}\,'' \quad O(m)$$

Remark: Computational cost : $\mathcal{O}(m^2) + \mathcal{O}(m)$ (multiplication + addition)

$$\mathcal{O}(m^2) + \mathcal{O}(m) \simeq \mathcal{O}(m^2)$$

Computational cost for FFT: Assume $n = 2^l$.

Let C_m = computational cost of F_m. Then $C_1 = 1$.

Claim: $C_{2m} = 2C_m + 3m$

Proof: **Step 2:** $\vec{y}' = F_m \vec{x}'$, $\vec{y}'' = F_m \vec{x}''$ $(= 2C_m)$

Step 3:

$$y_j = \vec{y}_j' + w_n^j \vec{y}_j''$$

$$y_{j+m} = \vec{y}_j' - w_n^j \vec{y}_j''$$

for $j = 0, 1, 2, \dots, m-1$

(= 1 multiplication + 1 addition + 1 substraction)

Total : $3m$

$$\therefore C_{2m} = 2C_m + 3m.$$

Now, $n = 2^{\ell}$.

$$\therefore \quad C_{2^{\ell}} = 2C_{2^{\ell-1}} + 3 \cdot 2^{\ell-1}.$$

$$\therefore \quad 2^{-\ell} C_{2^{\ell}} = 2^{-(\ell-1)} C_{2^{\ell-1}} + \frac{3}{2} = 2^{-(\ell-2)} C_{2^{\ell-2}} + 2\left(\frac{3}{2}\right)$$

$$= \vdots$$

$$= 2^{0} C_{2^{0}} + \ell\left(\frac{3}{2}\right) = 1 + \frac{3}{2}\ell$$

$$\therefore \quad C_{2^{\ell}} = \underbrace{2^{\ell}}_{n} + \frac{3}{2}\overbrace{\ell}^{\log_2 n}\underbrace{2^{\ell}}_{n} = n + \frac{3}{2} n \log_2 n = \mathcal{O}(n \log_2 n)$$

Butterfly diagram (Algorithmic visualization)

Consider F_4 (4x4 matrix).

[Recall: $\vec{y}_e = \vec{y}\,' = F_m \vec{x}\,'$.   Denote $\vec{x}\,' := \vec{x}_e$, $\vec{x}\,'' = \vec{x}_o$]

$\vec{y}_o = \vec{y}\,'' = F_m \vec{x}\,''$

$$\begin{tikzcd}
\vec{y}_e := F_2 \vec{x}_e = \begin{pmatrix} x_0 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \arrow[r] \arrow[dr] & \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\
\vec{y}_o := F_2 \vec{x}_o = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \arrow[ur, "\vec{\omega}_4"] \arrow[r, "-\vec{\omega}_4"'] & \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}
\end{tikzcd}$$

$$\vec{\omega}_4 = \begin{pmatrix} \omega_4^0 \\ \omega_4^1 \end{pmatrix}$$

Diagram means:

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \vec{y}_e + \vec{\omega}_4 \otimes \vec{y}_o$$

$$\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} = \vec{y}_o - \vec{\omega}_4 \otimes \vec{y}_o$$

} Depend on $\vec{y}_o$ and $\vec{y}_e$.

For $F_2\vec{x}_e$:

$$1 = F_1\vec{x}_{ee} = (x_0)$$
$$1 = F_1\vec{x}_{eo} = (x_2)$$
$$\vec{\omega}_2^{\,0}, \quad -\vec{\omega}_2^{\,0}$$
$$\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = F_2\vec{x}_e = \vec{y}_e$$

For $F_2\vec{x}_o$:

$$1 = F_1\vec{x}_{oe} = (x_1)$$
$$1 = F_1\vec{x}_{oo} = (x_3)$$
$$\vec{\omega}_2^{\,0}, \quad -\vec{\omega}_2^{\,0}$$
$$\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = F_2\vec{x}_o = \vec{y}_o$$

Remark:

$$\vec{x}_e = \begin{pmatrix} x_0 \\ x_2 \end{pmatrix}, \quad \vec{x}_o = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

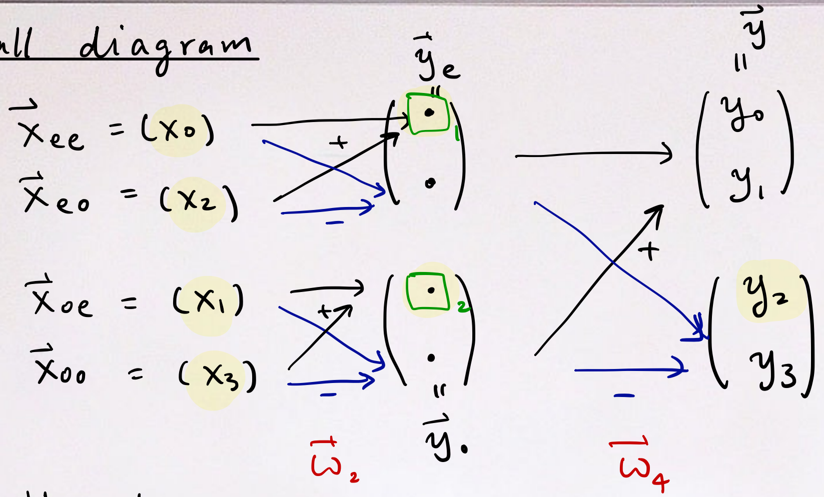
$$\vec{x}_{ee} = (x_0) \qquad \vec{x}_{oe} = (x_1)$$
$$\vec{x}_{eo} = (x_2) \qquad \vec{x}_{oo} = (x_3)$$

$$\vec{y}_e = \begin{pmatrix} x_0 + \vec{\omega}_2^{\,0} \otimes (x_2) \\ x_0 - \vec{\omega}_2^{\,0} \otimes (x_2) \end{pmatrix} = \begin{pmatrix} x_0 + x_2 \\ x_0 - x_2 \end{pmatrix}$$

$$\vec{y}_o = \begin{pmatrix} x_1 + \vec{\omega}_2^{\,0} \otimes (x_3) \\ x_1 - \vec{\omega}_2^{\,0} \otimes (x_3) \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_1 - x_3 \end{pmatrix}$$

$$\vec{\omega}_2^{\,0} = (\omega_2^0) = (1)$$

Overall diagram



(Butterfly diagram)

Using the diagram, find y_2.

$$y_2 = \square_1 - (\vec{\omega}_4)_0 \, \square_2 \quad ; \quad \square_1 = x_0 + x_2$$

$$= \square_1 - \square_2 \qquad\qquad \square_2 = x_1 + x_3$$

$$\therefore \; y_2 = x_0 + x_2 - (x_1 + x_3)$$

Iterative method to solve huge linear system

Recall: Numerical spectral method handles periodic functions.

Consider : $(*)\quad \frac{d^2u}{dx^2} = f \;,\; u(0) = A \;,\; u(1) = B$

$x \in [0,1]$.

Partition $[0,1]$ into $x_j = jh$ where $h = \frac{1}{n+1}$

Then: $(*)$ is discretized as:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i)$$

for all $i = 1, 2, \dots, n$

$u_i \overset{\text{def}}{=} u(x_i)$

or

$$\tilde{A}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} f(x_1) - \frac{A}{h^2} \\ f(x_2) \\ \vdots \\ f(x_n) - \frac{B}{h^2} \end{pmatrix}$$

where

$$\tilde{A} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$ has eigenvector $\overrightarrow{e^{ikx}}$

Question: How to solve BIG linear system?

Method 1: Gaussian elimination

Comp. cost : $\mathcal{O}(n^3)$

Sol : exact.

Method 2: LU factorization.

Decompose $A = LU$

lower upper

(If A is SPD, then $A = LL^T$ by Cholesky decomposition)

$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$ by solving $\begin{cases} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{cases}$ easy

Comp. cost : $\mathcal{O}(n^3)$

Sol : exact.

Goal: Develop iterative method: find a sequence $\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots$ such that $\vec{x}_k \to \vec{x}^*$ = sol. of $A\vec{x} = \vec{f}$ as $k \to \infty$.

Remark: We can stop when error is small enough.

Method: Splitting method

Consider a linear system $A\vec{x} = \vec{f}$ where $A \in M_{n\times n}$ (n is BIG)

Split A as follows: $A = N + (A - N) = N - \underbrace{(N - A)}_{P}$

$= N - P$

Then: $A\vec{x} = \vec{f} \iff (N-P)\vec{x} = \vec{f} \iff N\vec{x} = P\vec{x} + \vec{f}$

Develop an iterative scheme as follows:

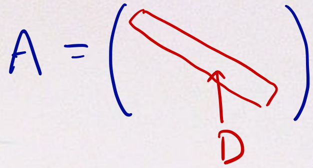
$$(*) \quad N\vec{x}^{n+1} = P\vec{x}^{n} + \vec{f}$$

If $\{\vec{x}^n\}_{n=1}^{\infty}$ converges, then it converges to the sol $\vec{x}^*$ of $A\vec{x} = \vec{f}$

Remark:

- N should be simple : easy to find inverse.
- N should have an inverse
- N should be "related to" A.
- N should be chosen such that $\{\vec{x}^n\}_{n=1}^{\infty}$ converges.

Splitting choice 1: Jacobi method



Take $N = D$ = diagonal part of A

Split A as $A = D - (D - A)$

Then: $A\vec{x} = \vec{f} \iff D\vec{x} - (D-A)\vec{x} = \vec{f}$

$$\iff D\vec{x} = (D-A)\vec{x} + \vec{f}$$

We can consider an iterative scheme such that:

$$D\vec{x}^{\,k+1} = (D-A)\vec{x}^{\,k} + \vec{f}$$

$$\iff \vec{x}^{\,k+1} = D^{-1}(D-A)\vec{x}^{\,k} + D^{-1}\bar{f}$$

Remark: All diagonal entries of A must be non-zero, such that D is non-singular.

This is equivalent to solving: (assume $A = (a_{ij})_{1 \le i,j \le n}$) $\vec{x}^k = \begin{pmatrix} x_1^k \\ \vdots \\ x_n^k \end{pmatrix}$

$$\begin{cases} a_{11} x_1^{k+1} + a_{12} x_2^k + a_{13} x_3^k + \dots + a_{1n} x_n^k = f_1 & (\text{for } x_1^{k+1}) \\ a_{21} x_1^k + a_{22} x_2^{k+1} + a_{23} x_3^k + \dots + a_{2n} x_n^k = f_2 & (\text{for } x_2^{k+1}) \\ \vdots \\ a_{n1} x_1^k + a_{n2} x_2^k + a_{n3} x_3^k + \dots + a_{nn} x_n^{k+1} = f_n & (\text{for } x_n^{k+1}) \end{cases}$$

<u>Example:</u> Consider: $\begin{pmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

Then: Jacobi method gives:

$$\vec{x}^{k+1} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 & -3 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \vec{x}^k + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Start with $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The sequence almost converge in 7 iteration to get: $\vec{x}_7 = \begin{pmatrix} -0.186 \\ 0.331 \\ -0.423 \end{pmatrix}$

Question:

- Does it always converge?
- Is the initialization important?