THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2024-2025
Assignment 1 Suggested Solution

1. Solve the following ODE using method of integrating factor

$$y' + \frac{1}{x}y - sin(x) = 0, \quad x > 0$$

with condition $y(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$.

Solution:

$$y' + \frac{1}{x}y - sin(x) = 0$$
$$y' + \frac{1}{x}y = sin(x)$$

Let $M(x) = e^{\int_{S_0}^{x} \frac{1}{s}ds} = e^{lnx - lnS_0} = \frac{x}{S_0}$.

$$\frac{d(M(x)y)}{dx} = M(x)\left(y' + \frac{1}{x}y\right) = M(x)sin(x) = \frac{x}{S_0}sin(x)$$

$$M(x)y = \frac{x}{S_0}y = \int \frac{x}{S_0}sin(x)dx = \frac{1}{S_0}(-xcos(x) + sin(x) + C)$$

$$y = -cos(x) + \frac{sin(x)}{x} + \frac{C}{x}$$

Substituting the initial condition,

$$y(\frac{\pi}{6}) = -cos(\frac{\pi}{6}) + \frac{sin(\frac{\pi}{6})}{\frac{\pi}{6}} + \frac{C}{\frac{\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{\frac{1}{2}}{\frac{\pi}{6}} + \frac{C}{\frac{\pi}{6}}$$

We have $C = -\frac{1}{2}$, and

$$y = -cos(x) + \frac{sin(x)}{x} - \frac{1}{2x}$$

2. Solve the following second order ODE using method of integrating factor

$$y'' - 4y - x^2 - 2x - 1 = 0$$

with conditions $y(0) = e^4 + e^{-4} - \frac{3}{8}$ and $y(2) = -\frac{3}{8}$.

Solution: We first solve for the general solutions. From the text, we have

$$y'' - 4y = 0$$

for the general solutions, Then we move the $4y$ to the right-hand side, and multiply on both sides by $\frac{dy}{dx}$. We have

$$\frac{d^2y}{dx^2}\frac{dy}{dx} = 4y\frac{dy}{dx}$$

which is equivalent to

$$\frac{d}{dx}((\frac{dy}{dx})^2) = \frac{d}{dx}(4y^2)$$

A possible solution of the above is:

$$(\frac{dy}{dx})^2 = 4y^2$$
$$\frac{dy}{dx} = \pm 2y$$

Using the integrating factor technique for the 1st order differential equation, we have:

$$y(x) = Ce^{\pm 2x}$$

for some constant C. Therefore, the general solution is $y(x) = \alpha_1 e^{-2x} + \alpha_2 e^{2x}$.

For the equation $y'' - 4y = x^2 + 2x + 1$, let $y_n(x) = ax^2 + bx + c$, and substitute it into the ODE, we have

$$\begin{aligned} 2a - 4(ax^2 + bx + c) &= x^2 + 2x + 1 \\ -4ax^2 - 4bx - 4c + 2a &= x^2 + 2x + 1 \end{aligned}$$

which indicates that $-4a = 1$, $-4b = 2$ and $-4c + 2a = 1$. Thus we have $a = -\frac{1}{4}$, $b = -\frac{1}{2}$ and $c = -\frac{3}{8}$.

Therefore, we have $y(x) = \alpha_1 e^{-2x} + \alpha_2 e^{2x} - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{8}$.

Substituting the initial conditions into the above equation, we obtain $\alpha_1 = e^4$ and $\alpha_2 = e^{-4}$.

3. Please show that

$$\int_0^{2\pi} \cos kx \cos mx \, dx = \begin{cases} 2\pi, \text{ if } k = m = 0 \\ \pi, \text{ if } k = m \neq 0 \\ 0, \text{ if } k \neq m \end{cases}$$

and that

$$\int_0^{2\pi} \sin kx \sin mx \, dx = \begin{cases} 0, \text{ if } k = m = 0 \\ \pi, \text{ if } k = m \neq 0 \\ 0, \text{ if } k \neq m \end{cases}$$

where m, k are non-negative integer.

Solution:

For cosine terms,

if $k = m = 0$,

$$\int_0^{2\pi} dx = 2\pi$$

if $k = m \neq 0$,

$$\begin{aligned} \int_0^{2\pi} \cos^2 kx dx &= \int_0^{2\pi} \frac{\cos 2kx + 1}{2} dx \\ &= \pi + \frac{1}{4k} \left[\sin 2kx\right]_0^{2\pi} \\ &= \pi \end{aligned}$$

if $k = m = 0$,

$$\begin{aligned} \int_0^{2\pi} \cos kx \cos mx dx &= \frac{1}{2} \int_0^{2\pi} \big(\cos(k+m)x + \cos(k-m)x\big) dx \\ &= \frac{1}{2} \left[\frac{1}{k+m} \sin(k+m)x + \frac{1}{k-m} \sin(k-m)x \right]_0^{2\pi} \\ &= 0 \end{aligned}$$

Sine terms are similar.

4. Find a possible Fourier series solution to the following differential equation

$$-2y''(x) + y(x) = f(x)$$

where $x \in (-L, L)$ and

$$f(x) = \begin{cases} 0, & if \ -L < x < 0, \\ \frac{1}{L}, & if \ 0 \le x < L \end{cases}$$

Solution:
Since the interval is $(-L, L)$, the family of trigonometric functions used here is $\{(\cos(\frac{2\pi}{2L}k\pi x), \sin(\frac{2\pi}{2L}kx)), k \in \mathbb{N}\}$ Computing the Fourier Series of f,

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_0^L \frac{1}{L}dx = \frac{1}{2L} \\ A_n &= \frac{1}{L}\int_0^L \frac{1}{L}\cos(\frac{\pi}{L}nx)dx \\ &= \frac{1}{L}\frac{1}{n\pi}\sin(\frac{\pi}{L}nx)\Big|_0^L = 0 \\ B_n &= \frac{1}{L}\int_0^L \frac{1}{L}\sin(\frac{\pi}{L}nx)dx \\ &= \frac{-1}{nL\pi}\cos(\frac{\pi}{L}nx)\Big|_0^L = \frac{1-(-1)^n}{nL\pi} \\ g(x) &= \frac{1}{2L} + \sum_{n=0}^{\infty}\frac{2}{(2n+1)L\pi}\sin(\frac{\pi}{L}(2n+1)x) \end{aligned}$$

According to the Fourier expansion form of $f(x)$, we suppose that

$$y(x) = a_0 + \sum_{n=0}^{\infty} b_{2n+1}\sin(\frac{\pi}{L}(2n+1)x)$$

By differentiating and comparing the fourier coefficients termwisely, we have

$$\begin{cases} a_0 = \frac{1}{2L}, \\ \left[2\left(\frac{\pi(2n+1)}{L}\right)^2 + 1\right] b_{2n+1} = \frac{2}{(2n+1)L\pi} \end{cases}$$

5. Solve the following PDE using Fourier series

$$\begin{cases} u_t - 2 = u_{xx}, & 0 < x < 2\pi, t > 0 \\ u_{xx}(t,0) = -2 = u_{xx}(t,2\pi), & t > 0 \\ u(0,x) = x - x^2, & 0 < x < 2\pi \end{cases}$$

Solution:
Following the hint, we construct a function $f(x)$ such that $f''(x) = -2$, and a natural choice is $f(x) = -x^2$.

Next, we suppose $u(t,x) = -x^2 + a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx) + \sum_{n=1}^{\infty} b_n(t)\sin(nx)$. By the boundary condition of the second derivative, $a_n(t) = 0$ for the terms $n \ge 1$.

Then, after differentiating $u(t,x)$ termwisely, we have

$$b_n'(t) = -n^2 b_n(t), \quad a_0'(t) = 0$$

implying that $a_0(t)$ is a constant function.

With an initial condition $-x^2 + a_0(0) + \sum_{n=1}^{\infty} b_n(0)\sin(nx) = x - x^2$, we can compute the sine series of $f(x) = x$ on $[0, 2\pi]$

$$\begin{aligned} a_0(0) &= \frac{1}{2\pi}\int_0^{2\pi} x dx = \pi \\ b_n(0) &= \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)dx \\ &= \frac{1}{\pi}\left(\frac{-1}{n}\cos(nx)x\Big|_0^{2\pi} - \int_0^{2\pi}\frac{-1}{n}\cos(nx)dx\right) \\ &= -\frac{2}{n} \end{aligned}$$

Last, solving ODE $b_n'(t) = -n^2 b_n(t)$, with $b_n(0) = -\frac{2}{n}$, we have

$$u(t,x) = -x^2 + \pi + \sum_{n=1}^{\infty} -\frac{2}{n}e^{-n^2 t}\sin(nx)$$

6. Let f be a 2π-periodic function and $f(t) = sinh(t) = \frac{e^t - e^{-t}}{2}$ for $t \in (-\pi, \pi)$, then find the real Fourier series and prove that

$$\frac{sinh(1)}{sinh(\pi)} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{k^2+1}\sin(k)$$

Solution:
Since f(x) is odd, its Fourier series expansion only contains sine terms, and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^x - e^{-x}}{2}\sin(nx)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^x - e^{-x}}{2}\frac{e^{inx} - e^{-inx}}{2i}dx \\ &= \frac{1}{4\pi i}\left(\frac{1}{1+ni}e^{(1+ni)x}\Big|_{-\pi}^{\pi} - \frac{1}{ni-1}e^{(ni-1)x}\Big|_{-\pi}^{\pi} - \frac{1}{1-ni}e^{(1-ni)x}\Big|_{-\pi}^{\pi} + \frac{1}{-(ni+1)}e^{-(ni+1)x}\Big|_{-\pi}^{\pi}\right) \\ &= \frac{1}{4\pi i}\left(\frac{(-1)^n}{1+ni}(e^{\pi} - e^{-\pi}) - \frac{(-1)^n}{ni-1}(e^{-\pi} - e^{\pi}) - \frac{(-1)^n}{1-ni}(e^{\pi} - e^{-\pi}) + \frac{(-1)^n}{-(ni+1)}(e^{-\pi} - e^{\pi})\right) \\ &= \frac{(-1)^{n-1}2n}{(1+n^2)\pi}\sinh(\pi) \end{aligned}$$

Therefore,

$$\sinh(x) = \frac{2\sinh(\pi)}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{1+k^2}\sin(kx), \text{ for } x \in (-\pi, \pi)$$

substitute $x = 1$ into the expansion, we prove the equation.