THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2024-2025 Term 1
Suggested Solutions of Homework Assignment 4

Q1

(a).

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = \frac{1}{2}.$$

(b). The cumulative distribution function of X is

$$F(t) = \begin{cases} \int_{10}^{t} \frac{10}{x^2} dx, & t \geq 10 \\ 0, & t < 10 \end{cases}$$
$$= \begin{cases} 1 - \frac{10}{t}, & t \geq 10 \\ 0, & t < 10 \end{cases}.$$

(c). Assume that the lifetimes of the electronic devices are independent. Let Y be the random variable of the number of devices that will function for at least 15 hours. Then Y has a binomial distribution with parameters $n = 6$ and p, where

$$p = P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} dx = \frac{2}{3}.$$

The required probability is

$$P(Y \geq 3) = 1 - \sum_{k=0}^{2} P(Y = k) = 1 - \sum_{k=0}^{2} \binom{6}{k} p^k (1-p)^{6-k} = \frac{656}{729} \approx 0.8999.$$

Q2

First, note that

$$1 = \int_{-\infty}^{\infty} f(x) dx = \int_{0}^{1} \left(a + bx^2\right) dx = a + \frac{1}{3} b.$$

Moreover, we have

$$\frac{3}{4} = E[X] = \int_{0}^{1} x \left(a + bx^2\right) dx = \frac{1}{2} a + \frac{1}{4} b.$$

By the above two equations, we have $a = 0$ and $b = 3$,

$$E[X^2] = \int_0^1 x^2 \left(0 + 3x^2\right) dx = \frac{3}{5}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 0.0375.$$

Q3

$P(1 < X < 3) = F(3-) - F(1) = F(3) - F(1) = (1 - 4^{-2}) - (1 - 2^{-2}) = \dfrac{3}{16}$.

Next, the expectation is

$$\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x f(x) dx \\
&= \int_0^{+\infty} x f(x) dx + \int_{-\infty}^{0} x f(x) dx \\
&= \int_0^{+\infty} \int_0^{+\infty} \chi_{[0,x]}(t) f(x) dt dx - \int_{-\infty}^{0} \int_{-\infty}^{0} \chi_{[x,0]}(t) f(x) dt dx \\
&= \int_0^{+\infty} \int_0^{+\infty} \chi_{[t,+\infty)}(x) f(x) dx dt - \int_{-\infty}^{0} \int_{-\infty}^{0} \chi_{(-\infty,t]}(x) f(x) dx dt \\
&= \int_0^{+\infty} \int_t^{+\infty} f(x) dx dt - \int_{-\infty}^{0} \int_{-\infty}^{t} f(x) dx dt \\
&= \int_0^{+\infty} (1 - F(t)) dt - \int_{-\infty}^{0} F(t) dt \\
&= \int_0^{+\infty} \frac{1}{(1+t)^2} dt \\
&= 1
\end{aligned}$$

Here, let A be a set in the real line where $\chi_A(x)$ is defined to be 1, if $x \in A$, and to be 0, if $x \notin A$.

Q4

The roots $x_{1,2} = \dfrac{-4Y \pm \sqrt{16Y^2 + 16(Y-6)}}{8}$ are real if and only if

$$16Y^2 + 16(Y - 6) \geq 0$$

So we need to find this probability

$$\begin{aligned}
P\left(16Y^2 + 16(Y-6) \geq 0\right) &= P(\{Y \geq 2\} \cup \{Y \leq -3\}) \\
&= P(Y \leq -3) + P(Y \geq 2) \\
&= 0 + \int_2^{\infty} \lambda e^{-\lambda x} dx \\
&= e^{-2\lambda} = e^{-6}
\end{aligned}$$

Q5

First, we use AB to denote the line segment. Let C be a point randomly chosen in AB. Let X be a random variable denoting the length of the line segment AC. We can see X is uniformly distributed on $[0, L]$. Also, the event the ratio of the shorter to the longer segment is less than $\frac{1}{4}$ can be represented as

$$E := \left\{\frac{X}{L-X} < \frac{1}{4}\right\} \cup \left\{\frac{L-X}{X} < \frac{1}{4}\right\}.$$

Then

$$\begin{aligned}
P(E) =& P(\{X < \frac{1}{5}L\} \cup \{X > \frac{4}{5}L\}) \\
=& \int_0^{\frac{1}{5}L} \frac{1}{L} dx + \int_{\frac{4}{5}L}^{L} \frac{1}{L} dx \\
=& \frac{2}{5}.
\end{aligned}$$

Q6

Assume that the annual rainfalls are independent from year to year. Let X be the random variable of annual rainfall. Then $X \sim N\left(40, 4^2\right)$.

$$P(X \leq 50) = P\left(\frac{X-40}{4} \leq 2.5\right) = \Phi(2.5) \approx 0.9938.$$

The required probability is $P(X \leq 50)^{10} \approx 0.9397$.

Q7

Denote $\frac{X-12}{\sqrt{4}}$ by Z. Then Z is a standard normal random variable.

$$0.1 = P\{X > c\} = P\left\{Z > \frac{c-12}{\sqrt{4}}\right\} = 1 - P\left\{Z \leq \frac{c-12}{2}\right\} = 1 - \Phi(\frac{c-12}{2}),$$

where Φ is the cumulative distribution function of the standard normal random variable.

Therefore, $c = 2 \cdot \Phi^{-1}(0.9) + 12$.

Q8

(a).

$$P(X > 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx = e^{-1}.$$

(b).

$$\begin{aligned} P(X \geq 10 \mid X > 9) &= \frac{P(\{X \geq 10\} \cap \{X > 9\})}{P(X > 9)} \\ &= \frac{P(X \geq 10)}{P(X > 9)} \\ &= \frac{\int_{10}^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx}{\int_9^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx} \\ &= \frac{e^{-10/2}}{e^{-9/2}} \\ &= e^{-1/2}. \end{aligned}$$

Q9

We assume that X is a continuous random variable with density $f(x)$.

$$\begin{aligned}
E\left[X^2\right] &= \int_0^k x^2 f(x)dx \le k\int_0^k xf(x)dx = kE[X] \\
\mathrm{Var}(X) &= E\left[X^2\right] - (E[X])^2 \\
&\le kE[X] - (E[X])^2 \\
&= -\left(E[X] - \frac{k}{2}\right)^2 + \frac{k^2}{4} \\
&\le \frac{k^2}{4}.
\end{aligned}$$

Q10

Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$. Recall that

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

Taking the derivative with respect to x twice, we get

$$f'(x) = \frac{-(x-\mu)}{\sqrt{2\pi}\sigma^3} e^{-(x-\mu)^2/2\sigma^2}$$

and

$$f''(x) = \frac{-\sigma^2 + (x-\mu)^2}{\sqrt{2\pi}\sigma^5} e^{-(x-\mu)^2/2\sigma^2}.$$

Thus $f''(x) = 0 \Leftrightarrow -\sigma^2 + (x-\mu)^2 = 0 \Leftrightarrow x = \mu - \sigma$ or $x = \mu + \sigma$, as claimed. So $\mu - \sigma$ and $\mu + \sigma$ are the points of inflection of this function.

Q11

(a). By integration by parts, we have

$$\begin{aligned} E\left[g'(Z)\right] &= \int_{-\infty}^{\infty} g'(x) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[g(x) e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} x g(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \end{aligned}$$

Here, we need the additional assumption that

$$\lim_{x \to \pm\infty} g(x) e^{-\frac{x^2}{2}} = 0.$$

Then we have $E\left[g'(Z)\right] = E[Zg(Z)]$.

(b). Put $g(x) = x^n$, for $n \geq 1$. Note that $\lim_{x \to \pm\infty} g(x) e^{-\frac{x^2}{2}} = 0$, so by (a), we have $E\left[g'(Z)\right] = E[Zg(Z)]$. Therefore, $E\left[Z^{n+1}\right] = E[Zg(Z)] = E\left[g'(Z)\right] = E\left[nZ^{n-1}\right] = nE\left[Z^{n-1}\right]$.

(c). By(b), we have $E\left(Z^4\right) = 3E\left(Z^2\right) = 3$.

Q12

Let F_X and F_{kX} be the distribution of X and kX respectively. Let f_X and f_{kX} be the density of X and kX respectively. For $t > 0$,

$$\begin{aligned} F_{kX}(t) = P(kX \leq t) = P(X \leq t/k) = F_X(t/k), \\ f_{kX}(t) = F'_{kX}(t) = \frac{1}{k} f_X(t/k) = \frac{\lambda}{k} e^{-\frac{\lambda}{k} t}. \end{aligned}$$

For $t < 0, F_{kX}(t) = 0$ and $f_{kX}(t) = 0$. Hence, kX is an exponential random variable with parameter λ/k.