THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2024-2025 Term 1
Midterm Exam

Q1 (20pts)

Suppose events A, B, and C are independent with probabilities 1/3, 1/2 and 1/6 respectively. Calculate the following probabilities:

1. $\mathbb{P}(A \cup B)$.

   Since A, B are independent, then

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) \\ &= \frac{1}{3} + \frac{1}{2} - \frac{1}{3} \times \frac{1}{2} \\ &= \frac{2}{3}.\end{aligned}$$

2. $\mathbb{P}(A \cup B \cup C)$.

   By the Inclusion-exclusion identity,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ &\quad - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ &\quad - \mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\ &= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} - \frac{1}{3} \times \frac{1}{2} - \frac{1}{3} \times \frac{1}{6} - \frac{1}{2} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{6} \\ &= \frac{13}{18} = 0.7222.\end{aligned}$$

3. $\mathbb{P}(\text{only } A \text{ occurs})$.

   Note that the event $E := \{ \text{ only } A \text{ occurs } \}$ can be written as

$$[(A \cup B \cup C) \backslash (B \cup C)].$$

Since B, C are independent, it follows that

$$\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) = \frac{7}{12}.$$

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}((A \cup B \cup C) \backslash (B \cup C)) \\
&= \mathbb{P}(A \cup B \cup C) - \mathbb{P}(B \cup C) \\
&= \frac{13}{18} - \frac{7}{12} \\
&= \frac{5}{36} = 0.1389.
\end{aligned}$$

4. $\mathbb{P}$(exactly one of the three events occurs).

   Note that the event $F :=$ { exactly one of the three events occurs } can be written as

   $$[(A \cup B \cup C) \backslash (B \cup C)] \cup [(A \cup B \cup C) \backslash (A \cup C)] \cup [(A \cup B \cup C) \backslash (A \cup B)].$$

   Because A, B, C are independent, then

   $$\begin{aligned}
   \mathbb{P}(B \cup C) &= \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) = \frac{7}{12}, \\
   \mathbb{P}(A \cup C) &= \mathbb{P}(A) + \mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(C) = \frac{4}{9}, \\
   \mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = \frac{2}{3}.
   \end{aligned}$$

   Since the events $(A \cup B \cup C) \backslash (B \cup C), (A \cup B \cup C) \backslash (A \cup C)$ and $(A \cup B \cup C) \backslash (A \cup B)$ are disjoint, then

   $$\begin{aligned}
   \mathbb{P}(F) &= \mathbb{P}((A \cup B \cup C) \backslash (B \cup C)) \\
   &\quad + \mathbb{P}((A \cup B \cup C) \backslash (A \cup C)) + \mathbb{P}((A \cup B \cup C) \backslash (A \cup B)) \\
   &= \mathbb{P}(A \cup B \cup C) - \mathbb{P}(B \cup C) \\
   &\quad + \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A \cup C) + \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A \cup B) \\
   &= \frac{13}{18} - \frac{7}{12} + \frac{13}{18} - \frac{4}{9} + \frac{13}{18} - \frac{2}{3} \\
   &= \frac{17}{36}.
   \end{aligned}$$

Q2 (10pts)

Suppose that E, F, G are independent events in a probability space. Let F^c denote the complements of F.

1. Show that E and F^c are independent.

   Since $E = (E \cap F) \cup (E \cap F^c)$, $E \cap F$ and $E \cap F^c$ are disjoint, then
   $$\begin{aligned} \mathbb{P}(E \cap F^c) &= \mathbb{P}(E) - \mathbb{P}(E \cap F) \\ &= \mathbb{P}(E) - \mathbb{P}(E)\mathbb{P}(F) \\ &= \mathbb{P}(E)(1 - \mathbb{P}(F)) \\ &= \mathbb{P}(E)\mathbb{P}(F^c) \end{aligned}$$
   then by the definition, E and F^c and independent.

2. Is $E \cup F$ independent of G ? Justify your answer.

   Since
   $$\begin{aligned} \mathbb{P}((E \cup F) \cap G) =& \mathbb{P}((E \cap G) \cup (F \cap G)) \\ & \text{(Inclusion-exclusion identity)} \\ =& \mathbb{P}(E \cap G) + \mathbb{P}(F \cap G) - \mathbb{P}((E \cap G) \cap (F \cap G)) \\ =& \mathbb{P}(E \cap G) + \mathbb{P}(F \cap G) - \mathbb{P}(E \cap F \cap G) \\ & (E, F, G \text{ are independent}) \\ =& \mathbb{P}(E)\mathbb{P}(G) + \mathbb{P}(F)\mathbb{P}(G) - \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G) \\ =& [\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E)\mathbb{P}(F)]\mathbb{P}(G) \\ =& [\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)]\mathbb{P}(G) \\ =& \mathbb{P}(E \cup F)\mathbb{P}(G) \end{aligned}$$
   then by the definition, $E \cup F$ and G are independent.

Q3 (10pts)

An urn contains n balls, one of which is special. We randomly select k balls from the urn, with each ball having an equal probability of being chosen. What is the probability that the special ball is chosen?

$$\mathbb{P}\{\text{ special ball is selected }\} = \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

Q4 (15pts)

There are two coins. When Coin 1 is flipped, it lands on head with probability 0.3; when Coin 2 is flipped, it lands on head with probability 0.7. We first choose 1 coin, where the probability of each coin being chosen is equal, and then flip it 10 times.

1. What is the probability that the coin lands on head for exactly 7 times of the 10 flips?

   Denote the event "first coin is flipped" by C_1, with C_2 defined similarly. Let X be the number of heads out of 10 tosses.

$$\begin{aligned}\mathbb{P}(X=7) &= \mathbb{P}(X=7 \mid C_1)\,\mathbb{P}(C_1) + \mathbb{P}(X=7 \mid C_2)\,\mathbb{P}(C_2)\\ &= \left[\binom{10}{7} 0.3^7 \cdot 0.7^3\right]\frac{1}{2} + \left[\binom{10}{7} 0.7^7 \cdot 0.3^3\right]\frac{1}{2}\\ &= \frac{34478703}{250000000}\\ &= 0.1379\end{aligned}$$

2. Given that the first of these 10 flips lands on head, what is the conditional probability that exactly 7 of the 10 flips land on heads?

   (b). By conditioning on the outcome of the first flip, we update the probability (now evenly split between coins 1 and 2) that coin 1 is being flipped. Let H_1 denote the event "the first flip is heads".

$$\begin{aligned}\mathbb{P}(C_1 \mid H_1) &= \frac{\mathbb{P}(H_1 \mid C_1)\,\mathbb{P}(C_1)}{\mathbb{P}(H_1 \mid C_1)\,\mathbb{P}(C_1) + \mathbb{P}(H_1 \mid C_2)\,\mathbb{P}(C_2)}\\ &= \frac{0.3 \times 0.5}{0.3 \times 0.5 + 0.7 \times 0.5}\\ &= 0.3\end{aligned}$$

   Our updated probabilities are now: $\mathbb{P}(C_1 \mid H_1) = 0.3$ and $\mathbb{P}(C_2 \mid H_1) = 0.7$. Then

$$
\begin{aligned}
&\mathbb{P}\left(X=7 \mid H_1\right) \\
&=\frac{\mathbb{P}\left(\{X=7\} \cap H_1\right)}{\mathbb{P}\left(H_1\right)} \\
&=\frac{\mathbb{P}\left(\{X=7\} \cap C_1 \cap H_1\right)+\mathbb{P}\left(\{X=7\} \cap C_2 \cap H_1\right)}{\mathbb{P}\left(H_1\right)} \\
&=\frac{\mathbb{P}\left(\{X=7\} \cap C_1 \cap H_1\right)}{\mathbb{P}\left(C_1 \cap H_1\right)} \cdot \frac{\mathbb{P}\left(C_1 \cap H_1\right)}{\mathbb{P}\left(H_1\right)}+\frac{\mathbb{P}\left(\{X=7\} \cap C_2 \cap H_1\right)}{\mathbb{P}\left(C_2 \cap H_1\right)} \cdot \frac{\mathbb{P}\left(C_2 \cap H_1\right)}{\mathbb{P}\left(H_1\right)} \\
&=\mathbb{P}\left(X=7 \mid C_1 \cap H_1\right) \cdot \mathbb{P}\left(C_1 \mid H_1\right)+\mathbb{P}\left(X=7 \mid C_2 \cap H_1\right) \cdot \mathbb{P}\left(C_2 \mid H_1\right) \\
&=\left[\binom{9}{6} 0.3^6 \cdot 0.7^3\right] 0.3+\left[\binom{9}{6} 0.7^6 \cdot 0.3^3\right] 0.7 \\
&=\frac{241350921}{1250000000} \\
&=0.1931
\end{aligned}
$$

Q5 (15pts)

Let X be a continuous random variable, having the density function given by

$$
f(x)=\begin{cases} c\left(x^2-x^3\right), & \text{if } 0<x<1, \\ 0, & \text{otherwise.} \end{cases}
$$

1. What is the value of the constant c ?

$$
1=\int_0^1 f(x) d x=\int_0^1 c\left(x^2-x^3\right) d x=\frac{c}{12},
$$

we have $c=12$.

2. Compute $\mathbb{E}[X]$.

$$
\mathbb{E}[X]=\int_0^1 x f(x) d x=\int_0^1 12\left(x^3-x^4\right) d x=\frac{3}{5}=0.6 .
$$

3. Compute Var(X).

$$\mathbb{E}[X^2] = \int_0^1 x^2 f(x)dx = \int_0^1 12(x^4 - x^5)dx = \frac{2}{5},$$
$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{5} - \left(\frac{3}{5}\right)^2 = \frac{1}{25}.$$

Q6 (10pts)

Let Z be a standard normal random variable.

1. Find the probability density function of $X = -2Z$.

$$\mathbb{E}[X] = -2\mathbb{E}[Z] = 0, \quad \text{Var}(X) = (-2)^2 \text{Var}(Z) = 4,$$

we have $X \sim N(0, 4)$. Then

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}, x \in \mathbb{R}.$$

2. Find $\mathbb{E}[Y]$ for $Y = (Z+1)^2$.

$$\mathbb{E}[Y] = \mathbb{E}\left[Z^2 + 2Z + 1\right] = \text{Var}(Z) + E[Z]^2 + 0 + 1 = 2.$$

Q7 (10pts)

Let X be a Poisson random variable with parameter λ. Show that

$$\mathbb{E}\big[X(X-1)\cdots(X-k+1)\big] = \lambda^k \quad \text{for } k = 2, 3, \ldots.$$

Fix some $k \geq 2$, then $x(x-1)\cdots(x-k+1) = 0$ for $x \in \{0, 1, \cdots, k-1\}$. For $x \geq k$,

$$\begin{aligned}
\mathbb{E}[X(X-1)\cdots(X-k+1)] &= \sum_{x=k}^{+\infty} \frac{x!}{(x-k)!}\mathbb{P}(X=x) \\
&= \sum_{x=k}^{+\infty} \frac{x!}{(x-k)!}\frac{e^{-\lambda}\lambda^x}{x!} \\
&= \lambda^k \sum_{x=k}^{+\infty} \frac{e^{-\lambda}\lambda^{x-k}}{(x-k)!} \\
&= \lambda^k \sum_{a=0}^{+\infty} \frac{e^{-\lambda}\lambda^a}{a!} \\
&= \lambda^k.
\end{aligned}$$

Q8 (10pts)

If Y is an exponential random variable with parameter $\lambda = 3$, what is the probability that the roots of the equation on x:

$$4x^2 + 4xY - Y + 6 = 0$$

are real?

The roots $x_{1,2} = \dfrac{-4Y \pm \sqrt{16Y^2 + 16(Y-6)}}{8}$ are real if and only if

$$16Y^2 + 16(Y-6) \geq 0$$

So we need to find this probability

$$\begin{aligned}
P\left(16Y^2 + 16(Y-6) \geq 0\right) &= P(\{Y \geq 2\} \cup \{Y \leq -3\}) \\
&= P(Y \leq -3) + P(Y \geq 2) \\
&= 0 + \int_2^{\infty} \lambda e^{-\lambda x} dx \\
&= e^{-2\lambda} = e^{-6}
\end{aligned}$$