Chap 5. Continuous r.v.'s.

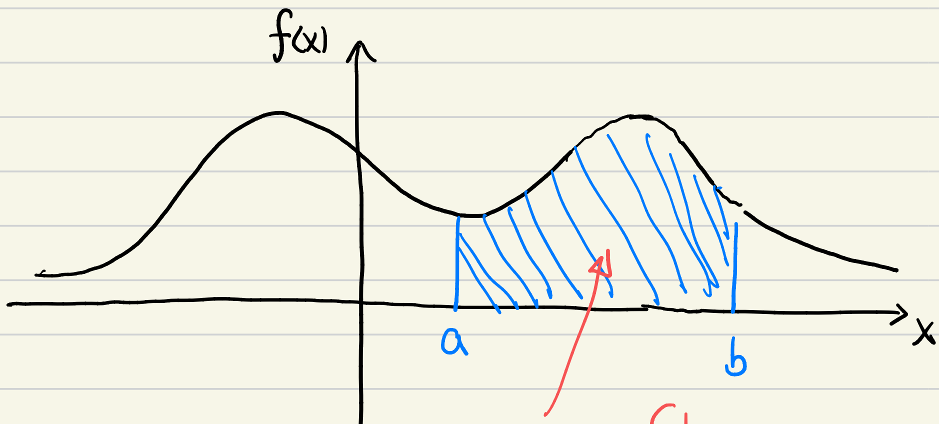
§ 5.1 Introduction.

Def. X is called a cont. r.v. (or absolute cont. r.v.) if there is a non-negative function defined on $(-\infty, \infty)$ such that

$$P\{X \in B\} = \int_B f(x)\,dx$$

for all "measurable" sets $B \subset (-\infty, \infty)$.

Remark: The measurable sets include all intervals, and the countable unions / intersections of intervals.



$$\text{Area} = \int_a^b f(x)\,dx$$

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

$$= \text{Area of the shaded region.}$$

- We call f the prob. density function (pdf) of X.

Example 2. Let X be a cts r.v. with pdf

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{if } x \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of C

(2) Find $P\{X \le 1\}$.

- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Since $P\{X \in (-\infty, \infty)\} = P\{S\} = 1$

Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^2 c\left(4x-2x^2\right)dx$$

$$= c\cdot\left(2x^2-\frac{2}{3}x^3\right)\Big|_0^2$$

$$= c\cdot\left(8-\frac{2}{3}\times 8\right) = \frac{8}{3}c.$$

Hence $c=\frac{3}{8}$.

$$P\{X\le 1\} = \int_{-\infty}^{1} f(x)\,dx$$

$$= \int_0^1 \frac{3}{8}\left(4x-2x^2\right)dx$$

$$= \frac{3}{8}\left(2x^2-\frac{2}{3}x^3\right)\Big|_0^1.$$

$$= \frac{1}{2}.$$

Exer. 1. Suppose X has a pdf

$$f(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of λ.

(2) Find $P\{X > 100\}$.

Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^{\infty} \lambda e^{-\frac{x}{100}}\,dx$$

$$= \lambda \cdot \left(-100\, e^{-\frac{x}{100}}\right)\Big|_0^{\infty}$$

$$= \lambda \cdot 100$$

Hence $\lambda = \frac{1}{100}$.

$$P\{X > 100\} = \int_{100}^{+\infty} f(x)\,dx$$

$$= \int_{100}^{+\infty} \frac{1}{100} e^{-\frac{x}{100}}\,dx$$

$$= -e^{-\frac{x}{100}}\Big|_{100}^{\infty}$$

$$= e^{-1}.$$

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§ 5.2 Expectation of a cts. r.v.

Recall that the expectation of a discrete r.v. X is defined by

$$E[X] = \sum x\,P\{X = x\},$$

where the summation is taken over all the possible values that X can take on.

We can not directly use this method to define the expectation of a cts r.v., since in the cts case, X will take on uncountably diff values, and moreover,

$$P\{X = x\} = 0 \text{ for all } x \in \mathbb{R}.$$

Def. Let X be a cts r.v. with pdf f.

Then we define

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Intuitive idea:

In the continuous case, we make a partition of $(-\infty, \infty)$ by $(x_n)_{n=-\infty}^{\infty}$ such that

$$x_{n+1} - x_n = \Delta x.$$

Then

$$\sum_n x_n \cdot P\{x_n < X \leq x_{n+1}\}$$

$$= \sum_n x_n \int_{x_n}^{x_n + \Delta x} f(x)\, dx$$

$$\approx \sum_n x_n f(x_n) \cdot \Delta x \longrightarrow \int_{-\infty}^{\infty} x f(x)\, dx$$

as $\Delta x \to 0$.

Example 2. X is said to be uniformly distributed on $[0,1]$ if it has the following density

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x \cdot 1\, dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}. \quad \square$$

Below we consider the expectation of functions of cts r.v.'s.

Prop 3. Let X be a cts r.v. with density f.
Let $g: \mathbb{R} \to \mathbb{R}$. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

We prove the above prop only in the case that $g \geq 0$.

To this end, we first prove the following.

Lem 4. Let Y be a non-negative cts r.v. Then

$$E[Y] = \int_0^{\infty} P\{Y > y\}\, dy.$$

To prove Lem 4, we need to use the following result:

$$\int_a^b \left(\int_c^d f(x,y)\, dx \right) dy = \int_c^d \left(\int_a^b f(x,y)\, dy \right) dx$$

when f is non-negative. This is a special version of the Fubini Theorem.

Pf of Lem 4. Let f be the density of Y.

Since Y is non-negative, $f(x)=0$ for $x \leq 0$.

So $E[Y] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x f(x)\, dx$. Observe

$$\int_0^{\infty} P\{Y > y\}\, dy$$

$$= \int_0^{\infty} \left(\int_y^{\infty} f(x)\, dx \right) dy$$

$$= \int_0^{\infty} \left(\int_y^{\infty} 1 \cdot f(x)\, dx + \int_0^{y} 0 \cdot f(x)\, dx \right) dy$$

$$= \int_0^{\infty} \left(\int_0^{\infty} g(x,y) f(x)\, dx \right) dy,$$

(where $g : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is defined by

$$g(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases}$$)

$$\overset{\text{Fubini}}{=\!=} \int_0^{\infty} \left(\int_0^{\infty} g(x,y) f(x)\, dy \right) dx$$

$$= \int_0^{\infty} f(x) \left(\int_0^{\infty} g(x,y)\, dy \right) dx$$

$$= \int_0^\infty f(x) \left(\int_0^x 1 \, dy + \int_x^\infty 0 \, dy \right) dx$$

$$= \int_0^\infty x f(x) \, dx$$

$$= E[Y]. \qquad \blacksquare$$

Proof of Prop. 3: By Lem 4,

$$E[g(X)] = \int_0^\infty P\{ g(X) > y \} \, dy.$$

Write $B := \{ x \in \mathbb{R} : g(x) > y \}$. Then

$$P\{ g(X) > y \} = P\{ X \in B \} = \int_B f(x) \, dx = \int_{\{x : g(x) > y\}} f(x) \, dx.$$

Hence

$$E[g(X)] = \int_0^\infty \int_{\{x : g(x) > y\}} f(x) \, dx \, dy$$

$$= \int_0^\infty \left(\int_{-\infty}^\infty h(x,y) \, f(x) \, dx \right) dy$$

where $h(x,y) = \begin{cases} 1 & \text{if } g(x) > y \\ 0 & \text{otherwise.} \end{cases}$

So by the Fubini Thm,

$$E[g(X)] = \int_{-\infty}^{\infty}\left(\int_0^{\infty} h(x,y)\, f(x)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_0^{\infty} h(x,y)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_0^{g(x)} h(x,y)\,dy + \int_{g(x)}^{\infty} h(x,y)\,dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_0^{g(x)} 1\, dy + \int_{g(x)}^{\infty} 0\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

$\boxtimes$.

Let X be a cts r.v. Define the Variance of X by

$$\mathrm{Var}(X) = E[(X-\mu)^2], \text{ where } \mu = E[X].$$

Prop. 1. $\mathrm{Var}(X) = E[X^2] - \mu^2$.

Pf. Let f be the density of X. Then

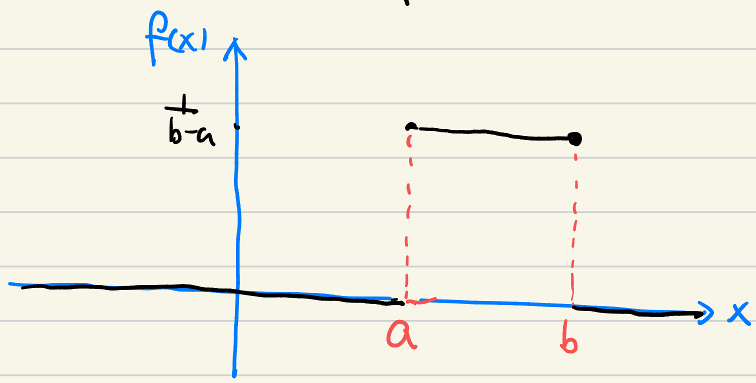
$$\begin{aligned}
\mathrm{Var}(X) &= E[(X-\mu)^2] \\
&= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu \int_{-\infty}^{\infty} x f(x)\,dx \\
&\qquad + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu\cdot\mu + \mu^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 \\
&= E[X^2] - \mu^2.
\end{aligned}$$

$\square$

§ 5.3. Uniform distributions.

Def. A cts X is said to be uniformly distributed on $[a,b]$ if it has a density

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$



We call X is a unif. r.v on $[a,b]$, or say X has a unif distribution on $[a,b]$.

Example 2. Calculate $E[X]$ and $V(X)$ for a unif r.v. X on $[a, b]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{\frac{1}{2}x^2}{b-a}\Big|_a^b = \frac{\frac{1}{2}(b^2-a^2)}{b-a}$$

$$= \frac{a+b}{2}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{3} \frac{b^3-a^3}{(b-a)} = \frac{a^2+ab+b^2}{3}$$

Hence

$$V(X) = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(a-b)^2}{12}.$$

- Def. (Cumulative distribution function).

Let X be a cts r.v with density f.

We define

$$F_X(b) = P\{X \leqslant b\}$$

$$= \int_{-\infty}^{b} f(x)\,dx.$$

Prop. 3. If f is cts at b, then

$$F_X'(b) = f(b).$$

Pf. Notice that for $u \in \mathbb{R}$, $u \neq 0$,

$$\frac{F_X(b+u) - F_X(b)}{u} = \frac{\int_{-\infty}^{b+u} f(x)\,dx - \int_{-\infty}^{b} f(x)\,dx}{u}$$

$$= \frac{1}{u}\int_b^{b+u} f(x)\,dx$$

Since f is cts at b, so

$f(x) - f(b)$ is close to 0

when x is close to b,

hence as $u \to 0$, $\frac{1}{u}\int_b^{b+u} f(x)\,dx \to f(b)$.

$\blacksquare$.

Example 4.

Let X be a unif r.v over $(0,1)$.

Find the density of X^2.

Solution : $F_{X^2}(b) = P\{X^2 \leq b\}$

$$= \begin{cases} P\{-\sqrt{b} \leq X \leq \sqrt{b}\}, & \text{if } b > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } b > 1, \\ \sqrt{b} & \text{if } b \in (0,1), \\ 0 & \text{if } b < 0. \end{cases}$$

Taking derivative we obtain

$$f_{X^2}(b) = \begin{cases} \frac{1}{2\sqrt{b}} & \text{if } b \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

$\square$