§ 4.6. Bernoulli r.v. and Binomial r.v.

(1) Bernoulli r.v.

Consider a random experiment, whose outcome can be classified as either a success, or a failure.

Define
$$X = \begin{cases} 1 & \text{if the outcome is a success,} \\ 0 & \text{if the outcome is a failure.} \end{cases}$$

Let $p = P\{X=1\}$, then $P\{X=0\} = 1-p$.

It has a prob. mass function: $\begin{cases} p(1) = p, \\ p(0) = 1-p. \end{cases}$

We call X a Bernoulli r.v. with parameter p.

- $E[X] = p$

  $E[X^2] = p$

  $V(X) = E[X^2] - E[X]^2 = p - p^2.$

(2) Binomial r.v.

Consider n independent trials, each of them results in either a success with prob p, or a failure with prob. $(1-p)$.

Let X = the number of the successes that appear in the n-trials.

We call X a Binomial r.v. with parameters (n, p).

- Example: $n=2$.

possible outcomes $\{(S,S), (S,F), (F,S), (F,F)\}$

$$P\{(S,S)\} = P(E_1 E_2) = P(E_1)P(E_2) = p \cdot p$$

where E_1 is the event that the outcome of the first trial is S

and E_2 is the event that the outcome of the second trial is S.

Similarly $P\{(S,F)\} = P\{(F,S)\} = p(1-p)$, $P\{(F,F)\} = (1-p)^2$.

Hence,

$$P\{X=1\} = P\{(S,F),(F,S)\} = 2 \cdot p(1-p).$$

- Prob. mass function for a general Binomial r.v. with paramters (n,p).

For $i=0,1,\cdots,n$, we have

$$\boxed{P\{X=i\} = \binom{n}{i}\cdot p^i(1-p)^{n-i}.}$$

Reason: The prob. of a special sequence of outcomes containing i successes and $(n-i)$ failures, is equal to $p^i(1-p)^{n-i}$

But there are in total $\binom{n}{i}$ such sequences, so

$$P\{X=i\} = \binom{n}{i}p^i(1-p)^{n-i}.$$

$$\left(\text{Recall } \binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}\right.$$

and

$$\left.(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i} \quad (\text{Binomial formula})\right)$$

Prop. Let X be a Binomial r.v. with parameters (n,p).

Let $k\geq 1$ be an integer. Then

$$E[X^k] = np\cdot E[(Y+1)^{k-1}],$$

where Y is a Binomial r.v. with parameters $(n-1,p)$.

Pf. By def,

$$E[X^k] = \sum_{i=0}^{n} i^k \cdot \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$\left(\text{using } i\binom{n}{i} = n\binom{n-1}{i-1}\right)$$

$$= \sum_{i=1}^{n} n \cdot i^{k-1} \binom{n-1}{i-1} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

Letting $j = i-1$

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$= np \cdot E[(Y+1)^{k-1}].$$

$\boxed{/\!/}$

Cor. $$E[X] = np \cdot E[(Y+1)^0] = np.$$

$$E[X^2] = np \cdot E[(Y+1)]$$

$$= np(E[Y]+1)$$

$$= np((n-1)p+1)$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= np((n-1)p+1) - (np)^2$$

$$= n(p-p^2).$$

§ 4.7 Poisson r.v.

Def. Let $\lambda > 0$. A r.v. X taking values in $\{0, 1, 2, \cdots\}$ is said to be a Poisson r.v. with parameter λ if

$$P\{X = i\} = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \quad i = 0, 1, \cdots$$

Remark. $$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}.$$

Hence $$\sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = 1.$$

A poisson r.v. can be used to approximate a binomial r.v with parameters (n, p) when n is large, p is small so that np is of moderate size.

Let X be a binomial r.v with parameters (n, p).

Let $np = \lambda$.

For $k = 0, 1, \cdots$,

$$P\{X = k\} = \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

$$= \frac{1 \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} \lambda^k \cdot \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\approx \frac{1}{k!} \lambda^k \cdot e^{-\lambda}$$

$\blacksquare$

Expected value and variance of Poisson r.v.

X —— Poisson r.v. with parameter λ.

$$P\{X=k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k=0, 1, \ldots$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

Letting $j=k-1$

$$= \sum_{j=0}^{\infty} e^{-\lambda} \cdot \lambda \cdot \frac{\lambda^j}{j!}$$

$$= \lambda \cdot \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}$$

$$= \lambda.$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} (k-1)\, e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$+ \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=2}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-2)!} + \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \lambda^2 + \lambda$$

So $\mathrm{Var}(X) = E[X^2] - E[X]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

§ 4.9 Expectation of sums of discrete r.v.'s.

Let $X_1, X_2, \cdots, X_n$ be discrete r.v.'s on the same sample space S.

Prop 1. $E[X_1 + \cdots + X_n] = \sum_{k=1}^{n} E[X_k]$.

We will prove the above result under an additional assumption that S is finite or countably infinite.

(The general case is referred to Theoretical Exer 4.36 in the text book).

Lem 2. Assume that S is finite or countably infinite.

Set $p(s) = P(\{s\})$ for $s \in S$.

Then for any r.v. X on S, we have

$$E[X] = \sum_{s \in S} X(s)\, p(s).$$

Pf. Suppose the distinct values of X are x_i, $i \geq 1$.

Let $S_i = \{ s \in S : X(s) = x_i \}$.

Then $S_1, S_2, \ldots,$ are a partition of S.

By definition,

$$E[X] = \sum_i x_i \, P\{X = x_i\}$$

$$= \sum_i x_i \, P(S_i)$$

$$= \sum_i x_i \sum_{s \in S_i} p(s)$$

$$= \sum_i \sum_{s \in S_i} x_i \, p(s)$$

$$= \sum_i \sum_{s \in S_i} X(s) \, p(s)$$

$$= \sum_{s \in S} X(s)\, p(s)$$

$\left(\text{because } S = \bigcup_i S_i\right.$ with the union being disjoint $\left.\right)$ $\blacksquare$

<u>Pf of Prop 1.</u>

By Lem 2,

$$E[X_1 + \cdots + X_n] = \sum_{s \in S} \left(X_1(s) + \cdots + X_n(s) \right) p(s)$$

$$= \left(\sum_{s \in S} X_1(s)\, p(s) \right) + \cdots + \left(\sum_{s \in S} X_n(s)\, p(s) \right)$$

$$= E[X_1] + \cdots + E[X_n].$$

$\blacksquare$.

§ 4.9. Cumulative distribution function.

Def. Let X be a discrete r.v. Define

$$F_X(b) = P\{X \leq b\}, \quad b \in \mathbb{R}.$$

We call F_X the cumulative distribution function (CDF) of X. We also write $F(b) = F_X(b)$.

Prop 3. (1) F is non-decreasing, that is

$$F(a) \leq F(b) \quad \text{if} \quad a < b.$$

(2) $\lim\limits_{b \to +\infty} F(b) = 1$.

(3) $\lim\limits_{b \to -\infty} F(b) = 0$.

(4) F is right continuous, i.e.

$$\lim_{b_n \downarrow b} F(b_n) = F(b).$$

(i.e. b_n tends to b from the RHS of b)

The prop. is based on the continuity property of probability.

Recall that if $E_n \nearrow E$, then $\lim P(E_n) = P(E)$

(i.e $E_{n+1} \supset E_n$, $E = \bigcup_{n=1}^{\infty} E_n$)

if $E_n \searrow E$ ($E_{n+1} \subset E_n$, $E = \bigcap_{n=1}^{\infty} E_n$)

then $P(E_n) \to P(E)$ as $n \to \infty$.

pf of Prop 3.

(1) Since if $a < b$, it follows that

$$\{X \leq a\} \subset \{X \leq b\}.$$

So $F(a) \leq F(b)$.

(2) If $b_n \nearrow \infty$,

then $\{X \leq b_n\} \nearrow \{X < \infty\} = S$

So $F(b_n) \to 1$ (by the continuity property of probability)

(3) If $b_n \searrow -\infty$, then

$$\{X \le b_n\} \searrow \{X = -\infty\} = \emptyset$$

So $F(b_n) \to 0$.

(4) If $b_n \searrow b$, then

$$\{X \le b_n\} \searrow \{X \le b\}.$$

So $\lim_{n\to\infty} F(b_n) = F(b)$.

Thus F is right continuous.

Remark: • In general, F is not left continuous.

Prop 4. $P\{X=b\} = F(b) - F(b-)$

Pf. Let $b_n \nearrow b$ with $b_n < b$.

Then $\{X \le b_n\} \nearrow \{X < b\}$.

So $P\{X \le b_n\} \to P\{X < b\}$.

Thus $P\{X<b\} = \lim_{n\to\infty} F(b_n) = F(b-)$.

It follows that

$$P\{X=b\} = P\{X \le b\} - P\{X<b\}$$
$$= F(b) - F(b-).$$

$\blacksquare$