Chap 3. Conditional probability and independence

§3.1 Conditional probability

Example 1: Let us roll two dices. Suppose the first die is a 3. Given this information, what is the prob. that the sum of 2 dices equals 8?

Sol: F — the event that the first die is 3

E — the event that the sum of 2 dices equals 8.

$$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$$

$$E = \{(i,j) \in \{1,2,3,4,5,6\}^2 : i+j = 8\}.$$

Prob of each outcome in F is $\frac{1}{6}$.

Hence the (conditional) prob of E given F is $\frac{1}{6}$.

Def. (Conditional prob.).

Let E, F be two events for a random experiment. Suppose $P(F) > 0$. Then the conditional prob. of E given F is

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Example 2: A coin is flipped twice. What is the conditional prob. that both flips land on heads given that the $\underset{\text{first}}{\wedge}$ flip lands on head?

Sol: Let F be the event that the first flip lands on head. That

$$F = \{(H,H), (H,T)\}.$$

Let E be the event that both flips land on heads.

$$E = \{(H,H)\}.$$

By def, $P(E|F) = \dfrac{P(EF)}{P(F)} = \dfrac{P\{(H,H)\}}{P\{(H,H),(H,T)\}}$

Notice that $S = \{(H,H), (H,T), (T,H), (T,T)\}$

Prop. (Multiplicative rule)

- $P(E_1 E_2) = P(E_1)\ P(E_2 | E_1)$
- $P(E_1 E_2 \cdots E_n)$

$$= P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 E_2) \cdots \cdot P(E_n | E_1 E_2 \cdots E_{n-1})$$

Pf. Since $P(E_2|E_1) = \dfrac{P(E_1 E_2)}{P(E_1)}$, so

$$P(E_1 E_2) = P(E_1) \cdot P(E_2|E_1).$$

To see the second identity,

$$RHS = P(E_1) \cdot \frac{P(E_1 E_2)}{P(E_1)} \cdot \frac{P(E_1 E_2 E_3)}{P(E_1 E_2)} \cdots \frac{P(E_1 \cdots E_n)}{P(E_1 \cdots E_{n-1})}$$

$$= P(E_1 \cdots E_n). \quad \square$$

Exer 1. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Solution. Let E denote the event that at least one die lands on 6, and let F denote the event that the dice land on different numbers.

Then

$$E = \{ (i,j) \in \{1,2,3,4,5,6\}^2 : i=6 \text{ or } j=6 \}$$

$$F = \{ (i,j) \in \{1,2,3,4,5,6\}^2 : i \neq j \}.$$

$$E \cap F = \{ (1,6), (2,6), \cdots, (5,6), (6,1), (6,2), \cdots, (6,5) \}$$

Notice that $\#(E \cap F) = 10$

$$\# F = 6 \times 5 = 30$$

Hence

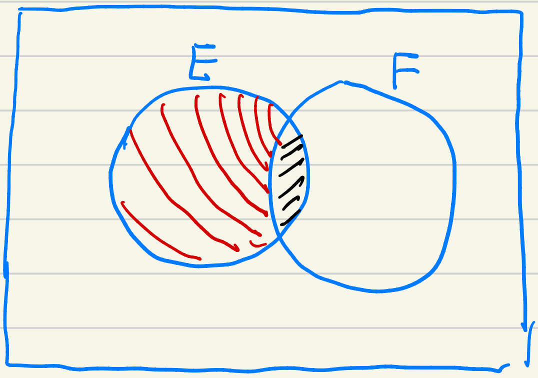
$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{\#(E\cap F) / \#S}{\# F / \# S}$$

$$= \frac{\#(E\cap F)}{\# F}$$

$$= \frac{10}{30} = \frac{1}{3}.$$

§ 3.2 Bayes' formula.

Let E, F be two events.



$$E = (E \cap F) \cup (E \cap F^c)$$

(black). (red)

Hence

$$P(E) = P(E \cap F) + P(E \cap F^c)$$

But $\quad P(E \cap F) = P(F) \cdot P(E|F),$

$$P(E \cap F^c) = P(F^c) \cdot P(E|F_c).$$

We obtain

$$P(E) = P(F) \cdot P(E|F) + P(F^c) \cdot P(E|F^c).$$

(Total probability formula).

Hence to determine the prob. of E,
we may first conduct the "conditioning"
upon whether or not the event F has
occured

Next we give a generenization of this formula.

Let $F_1, F_2, \cdots, F_n$ be a sequence of events
such that they are mutually exclusive,
and $\bigcup_{k=1}^{n} F_k = S$ (we say $F_1, \cdots, F_n$ are exhaustive)

Then we have

$$P(E) = \sum_{k=1}^{n} P(F_k) \cdot P(E \mid F_k).$$

pf: Notice that $E = \bigcup_{k=1}^{n} (E \cap F_k)$

(with disjoint union)

Hence

$$P(E) = \sum_{k=1}^{n} P(E \cap F_k)$$

$$= \sum_{k=1}^{n} P(F_k) P(E | F_k).$$

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Prop. (Bayes' formula).

Assume $F_1, \dots, F_n$ are mutually exclusive and exhaustive.

Then for any $1 \leq i \leq n$,

$$P(F_i | E) = \frac{P(F_i) \cdot P(E | F_i)}{\sum_{k=1}^{n} P(F_k) P(E | F_k)}$$

pf: $\sum_{k=1}^{n} P(F_k) P(E | F_k) = P(E)$

$P(F_i) \cdot P(E | F_i) = P(E F_i)$

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Example 3.

A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

(a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
(b) Given that a flashlight lasted more than 100 hours, what is the conditional probability that it was a type j flashlight, j = 1, 2, 3?

Solution: E —— the event that a random chosen flashlight will give more than 100 hours.

F_i $(i=1,2,3)$ —— the event that a random chosen flashlight is of type i.

We need to find out (a) $P(E)$

(b) $P(F_i \mid E)$.

From the conditions of the question, we know

$$P(E|F_1) = 0.7, \quad P(E|F_2) = 0.4$$

$$P(E|F_3) = 0.3.$$

$$P(F_1) = 0.2, \quad P(F_2) = 0.3, \quad P(F_3) = 0.5.$$

Hence $$P(E) = \sum_{i=1}^{3} P(F_i) P(E|F_i)$$

$$= 0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3$$

$$P(F_1|E) = \frac{P(F_1) \cdot P(E|F_1)}{P(E)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3}.$$

$$= \frac{14}{41}$$

Similarly

$$P(F_2|E) = \frac{12}{41}, \quad P(F_3|E) = \frac{15}{41}. \quad \boxed{}$$

Remark: Suppose $F \subset S$ is an event in a sample space with $P(F) > 0$.

Then $P(\cdot | F)$ is a probability on S.

(1) $P(S|F) = 1$.

(2) $0 \leq P(E|F) \leq 1$.

(3) $P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \Big| F\right) = \sum_{n=1}^{\infty} P(E_n|F)$,

if $E_1, E_2, \cdots$ are mutually exclusive.

(1), (2) are obvious. To see (3)

$$P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \Big| F\right) = \frac{P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \cap F\right)}{P(F)}$$

$$= \frac{P\left(\bigcup_{n=1}^{\infty} (E_n F)\right)}{P(F)}$$

$$= \sum_{n=1}^{\infty} \frac{P(E_n F)}{P(F)} \quad (\text{since } E_n F \text{ are disjoint})$$

$$= \sum_{n=1}^{\infty} P(E_n|F).$$

§3.3. Independent events.

Let E, F be two events. In general, knowing that F has occurred changes the chance of E's occurrence, that is, possibly $P(E|F) \neq P(E)$.

If $P(E|F) = P(E)$, we say E is independent of F.

Notice that

$$P(E|F) = P(E) \iff \frac{P(EF)}{P(F)} = P(E)$$

$$\iff P(EF) = P(E) \cdot P(F)$$

$$\iff P(F|E) = P(F)$$

Def. We say that E and F are independent if

$$P(EF) = P(E) \cdot P(F).$$

Example 2. A card is randomly chosen from a deck of 52 playing cards.

E — the event that the chosen card is an Ace "A"

F — the event that the chosen card is a spade. "♠"

Determine whether or not E and F are independent.

Solution:

$$P(E) = \frac{4}{52}, \quad P(F) = \frac{13}{52} = \frac{1}{4}.$$

$$P(EF) = \frac{1}{52} = P(E)\,P(F).$$

Hence E, F are independent.

Prop 3. If E and F are independent, then

(1) E and F^c are independent

(2) E^c and F^c are independent.

Pf. (1)

$$\begin{aligned}
P(E \cap F^c) &= P(E) - P(EF) \\
&= P(E) - P(E)\cdot P(F) \\
&= P(E)\big(1 - P(F)\big) \\
&= P(E)\,P(F^c),
\end{aligned}$$

Hence E, F^c are independent.

(2) can be obtained from (1).

- Independence of 3 or more events.

Def. We say 3 events E, F, G are independent if

(1) $P(EFG) = P(E)P(F)P(G)$.

(2) $P(EF) = P(E) \cdot P(F)$

$P(EG) = P(E)P(G)$

$P(FG) = P(F)P(G)$.

Def. Let $E_1, E_2, \cdots, E_n$ be a finite family of events,

Say $E_1, \cdots, E_n$ are independent if for any sub-collection $E_{i_1}, E_{i_2}, \cdots, E_{i_r}$ (with $i_1, \cdots, i_r$ being distinct),

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r}).$$

Def: We say an infinite family of events are independent if every finite subfamily of them is independent.

Def. (Independence of sub-experiments).

An experiment might consist of some sub-experiments,

For instance, the experiment that rolling a coining continuously consists of a sequence of sub-experiments, where the n-th subexperiment is the n-th toll of the coin, $n = 1, 2, \cdots$.

We say these sub-experiments are independent if

E_1 E_2, $\cdots$, E_n are independent

whenever E_i is an event whose occurence depends only on the i-th sub-experiment.

These sub-experiments are said to be trials if the set of possible outcomes of each sub-experiment are the same.