Prop.[1] (Continuity of Probability)

① $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \subset E_2 \subset \cdots$

② $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \supset E_2 \supset \cdots$

Pf. We first prove ①.

Write $F_1 = E_1$

$F_2 = E_2 \setminus E_1$

$\cdots$

$F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$

$\cdots$

Then $F_1, \cdots, F_n, \cdots$ are mutually exclusive.

and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$

$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

Hence

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n) \quad \text{(since } (F_n) \text{ are mutually disjoint)}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(F_k)$$

$$= \lim_{n\to\infty} P(F_1 \cup \cdots \cup F_n) \quad \text{(since } F_1, \cdots, F_n \text{ are disjoint)}$$

$$= \lim_{n\to\infty} P(E_n).$$

Next we prove ②.

Notice that

$$E_1^c \subset E_2^c \subset \cdots$$

By ①,

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c).$$

But

$$\text{LHS} = 1 - P\left(\bigcap_{n=1}^{\infty} E_n\right),$$

$$\text{RHS} = \lim_{n\to\infty} 1 - P(E_n).$$

This implies that $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$.

Example 2. If $P(E) = 0.8$, $P(F) = 0.9$

Show that $P(E \cap F) \geq 0.7$.

Pf. Recall

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Hence

$$\begin{aligned} P(E \cap F) &= P(E) + P(F) - P(E \cup F) \\ &= 0.8 + 0.9 - P(E \cup F) \\ &\geq 0.8 + 0.9 - 1 = 0.7. \quad \blacksquare \end{aligned}$$

Example 3.

If $P(E) = 0.8$, $P(F) = 0.9$, $P(E \cap F) = 0.75$

find the probability that exactly one of E and F occurs.

Solution: Let H denote the event that exactly one of E and F occurs.

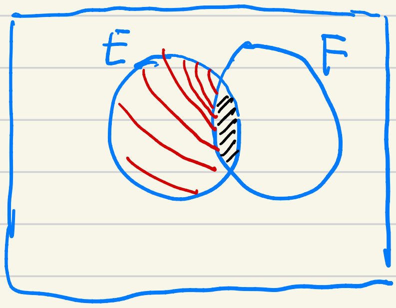
Then

$$H = (E \setminus F) \cup (F \setminus E)$$

(disjoint union)

Hence $P(H) = P(E \setminus F) + P(F \setminus E)$.

Notice that $E = (E \setminus F) \cap (E \cap F)$



Hence $P(E) = P(E \setminus F) + P(E \cap F)$

It follows that

$$P(E \setminus F) = P(E) - P(E \cap F)$$
$$= 0.8 - 0.75$$
$$= 0.05.$$

Similarly,

$$P(F \setminus E) = P(F) - P(E \cap F)$$
$$= 0.9 - 0.75$$
$$= 0.15$$

Hence

$$P(H) = P(E \setminus F) + P(F \setminus E)$$
$$= 0.05 + 0.15$$
$$= 0.20.$$

$\boxtimes$

§2.6 Sample space having equally likely outcomes.

In many experiments, it is natural to assume that all outcomes have the same chance to occur.

In this case,

$$P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S} = \frac{\# E}{\# S}.$$

Example 1. If two dices are rolled, what is the prob. that the sum of two outcomes is equal to 8 ?

Solution: Let E be the event that the sum of two outcomes is equal to 6. Then

$$E = \{ (i,j) : \quad i, j \in \{1, 2, \cdots, 6\}, \ i+j=8 \}$$
$$= \{ (2,6), (3,5), (4,4), (5,3), (6,2) \}$$

and

$$S = \{ (i,j) : \quad i, j \in \{1, 2, \cdots, 6\} \}$$

Hence $P(E) = \dfrac{\# E}{\# S} = \dfrac{5}{36}$. $\square$

Exer 2.

A committee of 5 is to be selected from a group of 6 men and 9 women. If the
selection is made randomly, what is the probability that the committee consists of 3
men and 2 women?

Solution: Let E denote the event that the selected committee consists of 3 men and 2 women.

Let S be whole sample space.

Then

$$\# S = \binom{15}{5},$$

$$\# E = \binom{6}{3} \cdot \binom{9}{2}.$$

Hence $$P(E) = \frac{\# E}{\# S} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}.$$

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}. \qquad \left(n! = n \times (n-1) \times \cdots 1 \right.$$
$$\left. 0! = 1 \right)$$

$\boxed{\text{//}}$

Exer 3.

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that
(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?

(a)

Solution: Let E be the event that one of the players receives all 13 spades.

Let E_i be the event that the i-th player receives all 13 spades, $i=1,2,3,4$.

$$E = \bigcup_{i=1}^{4} E_i, \qquad E_1, \cdots, E_4 \text{ are mutually exclusive.}$$

So $P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$.

$$\# E_1 = \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}$$

Similarly, $\# E_2 = \# E_3 = \# E_4 = \# E_1$.

$$\# S = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

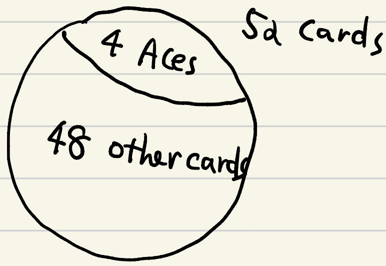
Hence
$$P(E_1) = \frac{\# E_1}{\# S} = \frac{\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = \frac{1}{\binom{52}{13}}$$

So is $P(E_i)$, $i=2,3,4$.

$$P(E) = \sum_{i=1}^{4} P(E_i) = \frac{4}{\binom{52}{13}}.$$

(b)

Let F be the event that each player receives an Ace.



$$\# F = \underline{\binom{4}{1} \cdot \binom{48}{12}} \cdot \binom{3}{1} \cdot \binom{36}{12} \cdot \underline{\binom{2}{1}\binom{24}{12}}$$

Hence $$P(F) = \frac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}}.$$

Exer 4. A deck of 52 cards in dealt out. What is the probability that the first ace occurs in the 14th card.

Solution: Let E denote the event that the first ace occurs in the 14 th card. Let S denote the sample space.

Then $\# S = 52!$

$$\# E = 48 \times 47 \times \cdots \times 36 \times 4 \times (38!)$$

Hence

$$P(E) = \frac{\# E}{\# S} = \frac{48 \times 47 \times \cdots \times 36 \times 4}{52 \times 51 \times \cdots \times 39}.$$