MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 6

7.2-10 If f and g are continuous on $[a,b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Solution. Suppose $f(x) \neq g(x)$ for any $x \in [a,b]$. Then the Intermediate Value Theorem implies that either $f - g > 0$ or $g - f > 0$ on $[a,b]$. Together with $\displaystyle\int_a^b (f-g) = \int_a^b f - \int_a^b g = 0$, Exercise 7.2-8 (see HW5) implies that $f - g = 0$ on $[a,b]$, which contradicts the assumption at the beginning. □

7.2-12 Show that $g(x) := \sin(1/x)$ for $x \in (0,1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0,1]$.

Solution. Clearly $|g(x)| \leq 1$ for all $x \in [0,1]$.

Let $\varepsilon > 0$. Choose $c \in (0,1)$ such that $c < \varepsilon/4$. On $[c,1]$, $g(x) = \sin(1/x)$ is continuous, and hence $g \in \mathcal{R}[c,1]$ by Proposition 2.13. By Theorem 2.10, there is a partition $P : c = x_1 < \cdots < x_n = 1$ on $[c,1]$ such that

$$0 \leq U(g,P) - L(g,P) = \sum_{i=1}^{n} \omega_i(g,P)\Delta x_i < \varepsilon/2,$$

where $\omega_i(g,P) := \sup\{|g(x) - g(x')| : x, x' \in [x_{i-1}, x_i]\}$. Now $P' : 0 =: x_0 < x_1 = c < x_2 < \cdots < x_n = 1$ is a partition on $[0,1]$ that satisfies

$$\begin{aligned} 0 \leq U(g,P') - L(g,P') &= \sum_{i=1}^{n} \omega_i(g,P')\Delta x_i \\ &= \sup\{|g(x) - g(x')| : x, x' \in [0,c]\}(c-0) + \sum_{i=2}^{n} \omega_i(g,P)\Delta x_i \\ &< 2(\varepsilon/4) + \varepsilon/2 = \varepsilon. \end{aligned}$$

By Theorem 2.10 again, $g \in \mathcal{R}[0,1]$. □

7.2-15 If f is bounded and there is a finite set E such that f is continuous at every point of $[a,b]\backslash E$, show that $f \in \mathcal{R}[a,b]$.

Solution. Let $\varepsilon > 0$ be given. Set $M = \sup|f(x)|$. Since E is finite, we can cover E by finitely many disjoint intervals $[u_j, v_j] \subseteq [a,b]$ such that $\sum |v_j - u_j| < \varepsilon$. Furthermore, we can place these intervals in such a way that every point of $E \cap (a,b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from $[a,b]$. The remaining set K is compact. Hence f is uniformly continuous on K, and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s, t \in K$ and $|s - t| < \delta$.

Now form a partition $P : a = x_0 < x_1 < \cdots < x_n = b$ such that

- every u_j and v_j occur in P,
- no point of any segment (u_j, v_j) occurs in P,
- $\Delta x_i := x_i - x_{i-1} < \delta$ if x_{i-1} is not one of the u_j.

Note that if $[x_{i-1}, x_i] \cap S = \emptyset$, then $\omega_i(f, P) \leq \varepsilon$; while if $[x_{i-1}, x_i] \cap S \neq \emptyset$, then $[x_{i-1}, x_i] = [u_j, v_j]$ for some j and $\omega_i(f, P) \leq 2M$. Hence,

$$\begin{aligned}\sum_{i=1}^{n} \omega_i(f, P)\Delta x_i &= \sum_{i:[x_{i-1},x_i]\cap S=\emptyset} \omega_i(f, P)\Delta x_i + \sum_{i:[x_{i-1},x_i]\cap S\neq\emptyset} \omega_i(f, P)\Delta x_i \\ &\leq \varepsilon \sum_{i:[x_{i-1},x_i]\cap S=\emptyset} \Delta x_i + 2M \sum_{j} (v_j - u_j) \\ &\leq \varepsilon(b - a) + 2M\varepsilon.\end{aligned}$$

By Theorem 2.10, $f \in \mathcal{R}[a, b]$.

□