MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 5

7.2-2 Consider the function h defined by $h(x) := x+1$ for $x \in [0,1]$ rational, and $h(x) := 0$ for $x \in [0,1]$ irrational. Show that h is not Riemann integrable.

Solution. By the density of rational and irrational numbers,

$$\sup_{a\le x\le b} h(x) = b+1, \qquad \text{and} \qquad \inf_{a\le x\le b} h(x) = 0.$$

For any partition $P : 0 = x_0 < x_1 < \cdots < x_n = 1$, we have

$$U(h,P) = \sum_{i=1}^{n}(x_i+1)(x_i - x_{i-1}) \ge \sum_{i=1}^{n}(x_i - x_{i-1}) = 1,$$

and

$$L(h,P) = \sum_{i=1}^{n}(0)(x_i - x_{i-1}) = 0.$$

Thus

$$\underline{\int_0^1} h = 0 < 1 \le \overline{\int_0^1} h.$$

Therefore h is not Riemann integrable. □

7.2-8 Suppose that f is continuous on $[a,b]$, that $f(x) \ge 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution. Suppose $f(x_0) > 0$ for some $x_0 \in [a,b]$. By the continuity of f, there is a nondegenerate subinterval $[c,d] \subseteq [a,b]$ such that $x_0 \in [c,d]$ and $f(x) > f(x_0)/2$ for all $x \in [c,d]$. Now if P is a partition of $[a,b]$ with $[c,d]$ as a subinterval, we have

$$L(f,P) \ge \frac{f(x_0)}{2}(d-c) =: m > 0.$$

Since $\int_a^b f = 0$, we have

$$0 = \int_a^b f = \underline{\int_a^b} f \ge m > 0,$$

which is a contradiction. Therefore, $f(x) = 0$ for all $x \in [a,b]$. □

7.2-9 Show that the continuity hypothesis in the preceding exercise cannot be dropped.

Solution. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \le 1. \end{cases}$

Clearly f is not continuous at 0.

For any partition $P : 0 = x_0 < x_1 < \cdots < x_n = 1$, we have

$$U(f,P) = (1)(x_1 - x_0) + \sum_{k=2}^{n}(0)(x_i - x_{i-1}) = x_1,$$

and

$$L(f,P) = \sum_{k=1}^{n} (0)(x_i - x_{i-1}) = 0.$$

Thus

$$\underline{\int_0^1} f = \overline{\int_0^1} f = 0.$$

Therefore f is Riemann integrable on $[0,1]$ with $\displaystyle\int_0^1 f = 0$. However, $f(0) = 1 \neq 0$. $\square$