MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 4

6.4-10 Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

Solution. First, we show that $\lim_{x\to 0} h(x)/x^k = 0$ for any $k \in \mathbb{N}$. By successive application of L'Hospital's Rule,

$$\lim_{y\to+\infty} \frac{y^k}{e^y} = \lim_{y\to+\infty} \frac{ky^{k-1}}{e^y} = \cdots = \lim_{y\to+\infty} \frac{k!}{e^y} = 0 \quad \text{for any } k \in \mathbb{N}.$$

Let $y = 1/x^2$. Then $y \to +\infty$ as $x \to 0$. Hence, for any $k \in \mathbb{N}$,

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{(1/x^2)^k}{e^{1/x^2}} \cdot x^k = 0. \tag{$*$}$$

Next, we calculate $h^{(n)}(x)$ for $x \neq 0$. Clearly $h(x) = e^{-1/x^2}$ is infinitely differentiable for $x \neq 0$. By applying Leibniz's rule to $h'(x) = \frac{2}{x^3}e^{-1/x^2} = \frac{2}{x^3}h(x)$, we have

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(n-k+2)!}{x^{n-k+3}} h^{(k)}(x) \tag{$**$}$$

for any $x \neq 0$ and integer $n \geq 0$.

Now, we prove by induction on n that

(i) $\lim_{x\to 0} \frac{h^{(n)}(x)}{x^m}$ for any $m \in \mathbb{N}$;

(ii) $h^{(n)}(0) = 0$.

The case $n = 0$ follows immediately from $(*)$. Suppose (i) and (ii) are true for n. Then $(**)$ gives

$$\lim_{x\to 0} \frac{h^{(n+1)}(x)}{x^m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (n-k+2)! \left(\lim_{x\to 0} \frac{h^{(k)}(x)}{x^{n-k+3+m}} \right) = 0.$$

Moreover,

$$h^{(n+1)}(0) = \lim_{x\to 0} \frac{h^{(n)}(x) - h^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(n)}(x)}{x} = 0.$$

This completes the induction.

Finally, the remainder term in Taylor's Theorem is given by

$$R_n(x) = h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} x^k = h(x),$$

and so $\lim_{x\to 0} R_n(x) = h(x) \neq 0$ for $x \neq 0$. □

6.4-15 Let f be continuous on $[a, b]$ and assume that the second derivative f'' exists on (a, b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

Solution. Applying the Mean Value Theorem to f on $[a, x_0]$, there exists $c_1 \in (a, x_0)$ such that

$$\frac{f(x_0) - f(a)}{x_0 - a} = f'(c_1).$$

Applying the Mean Value Theorem to f on $[x_0, b]$, there exists $c_2 \in (x_0, b)$ such that

$$\frac{f(b) - f(x_0))}{b - x_0} = f'(c_2).$$

By the assumption, the line segment joining $(a, f(a))$ and $(x_0, f(x_0))$ has the same slope as the line segment joining $(x_0, f(x_0))$ and $(b, f(b))$, thus

$$f'(c_1) = \frac{f(x_0) - f(a)}{x_0 - a} = \frac{f(b) - f(x_0))}{b - x_0} = f'(c_2).$$

Note $a < c_1 < c_2 < b$. Since f'' exists on (a, b), we have that f' is continuous and differentiable on $[c_1, c_2]$. By the Mean Value Theorem again, there exists $c \in (c_1, c_2) \subseteq (a, b)$ such that

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = 0.$$

□