MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 3

6.3-2 In addition to the supposition of the preceding exercise, let $g(x) > 0$ for $x \in [a,b]$, $x \neq c$. If $A > 0$ and $B = 0$, prove that we must have $\lim_{x\to c} f(x)/g(x) = \infty$. If $A < 0$ and $B = 0$, prove that we must have $\lim_{x\to c} f(x)/g(x) = -\infty$.

Solution. Suppose $A > 0$ and $B = 0$. Let $\alpha > 0$. By the assumption, there exists $\delta > 0$ such that for all $x \in [a,b] \cap V_\delta(c)\backslash\{c\}$, we have

$$f(x) > A/2 > 0, \quad \text{and} \quad 0 < g(x) < \frac{A/2}{\alpha},$$

which implies that

$$\frac{f(x)}{g(x)} > \alpha.$$

Therefore $\lim_{x\to c} f(x)/g(x) = \infty$.

If $A < 0$ and $B = 0$, the limit follows from above by considering $-f$. □

6.3-5 Let $f(x) := x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) := 0$, and let $g(x) := \sin x$ for $x \in \mathbb{R}$. Show that $\lim_{x\to 0} f(x)/g(x) = 0$ but $\lim_{x\to 0} f'(x)/g'(x)$ does not exist.

Solution. Note that, for $x \neq 0$,

$$\left|\frac{f(x)}{g(x)}\right| = |x|\,|\sin(1/x)|\left|\frac{x}{\sin x}\right| \leq |x|.$$

It then follows from Squeeze theorem that $\lim_{x\to 0} f(x)/g(x) = 0$.

On the other hand, $\lim_{x\to 0} f'(x)/g'(x) = \lim_{x\to 0} \frac{2x\sin(1/x)-\cos(1/x)}{\cos x}$ does not exist by applying sequential criterion to the sequences (x_n), (y_n), where

$$x_n := \frac{1}{2n\pi} \quad \text{and} \quad y_n := \frac{1}{(2n+1)\pi}.$$

□

6.4-4 Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Solution. Let $f(x) = \sqrt{1+x}$. Then, for any $x > -1$,

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f''(x) = -\frac{1}{4(1+x)^{3/2}}, \quad f'''(x) = \frac{3}{8(1+x)^{5/2}}.$$

Fix $x > 0$. By Taylor's Theorem, there exists $c_1 \in (0,x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(c_1)}{2!}(x-0)^2 \\ &= 1 + \frac{1}{2}x - \frac{1}{8(1+c_1)^{3/2}}x^2. \end{aligned}$$

Since $-\frac{1}{8(1+c_1)^{3/2}}x^2 < 0$, we have $\sqrt{1+x} \le 1 + \frac{1}{2}x$.

Similarly, there exists $c_2 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c_2)}{3!}(x-0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{5/2}}x^3. \end{aligned}$$

Since $\frac{1}{16(1+c_2)^{5/2}}x^3 > 0$, we have $1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x}$. $\square$