MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 2

6.2-5 Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$. [$H$int: Show that $f(x) := x^{1/n} - (x-1)^{1/n}$ is decreasing for $x \geq 1$, and evaluate f at 1 and a/b.]

Solution. Let $f(t) = t^{1/n} - (t-1)^{1/n}$ for $t \geq 1$. Then

$$f'(t) = \frac{1}{n}t^{1/n-1} - \frac{1}{n}(t-1)^{1/n-1} < 0 \qquad \text{for } t > 1.$$

For $x > 1$, since f is continuous on $[1, x]$ and differentiable on $(1, x)$, the Mean Value Theorem infers that there is $c_x \in (1, x)$ such that

$$f(x) - f(1) = f'(c_x)(x-1),$$

and so $f(x) < f(1) = 1$. Putting $x = \frac{a}{b} > 1$, we have $f(\frac{a}{b}) < 1$, which yields

$$a^{1/n} - b^{1/n} < (a-b)^{1/n}.$$

□

6.2-7 Use the Mean Value Theorem to prove that $(x-1)/x < \ln x < x - 1$ for $x > 1$. [Hint: Use that fact that $D \ln x = 1/x$ for $x > 0$.]

Solution. See Homework 1. □

6.2-15 Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I.

Solution. Since f' is bounded on I, there is $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. If $u, v \in I$, the Mean Value Theorem infers that there is c between u and v such that

$$f(u) - f(v) = f'(c)(u-v),$$

and thus

$$|f(u) - f(v)| = |f'(c)||u-v| \leq M|u-v|.$$

Hence f satisfies a Lipschitz condition on I. □