MATH 2060A Mathematical Analysis II
2024-25 Term 1
Suggested Solution to Homework 1

6.1-4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$.

Solution. Note that

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} \frac{x^2 - 0}{x} = x & \text{if } x \in \mathbb{Q}\backslash\{0\}, \\ \frac{0-0}{x} = 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

So,

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq |x| \quad \text{for any } x \in \mathbb{R}\backslash\{0\}.$$

Since $\lim_{x\to 0} |x| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. Therefore f is differentiable at $x = 0$ and $f'(0) = 0$.

□

6.1-10 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) := 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

Solution. If we use the fact that $D \sin x = \cos x$ for all $x \in \mathbb{R}$ and apply the Product Rule 6.1.3(c) and the Chain Rule 6.1.6, we obtain

$$g'(x) = 2x \sin(1/x^2) - 2x^{-1} \cos(1/x^2) \qquad \text{for } x \neq 0.$$

If $x = 0$, the definition of derivative yields

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x^2)}{x} = \lim_{x\to 0} x \sin(1/x^2) = 0,$$

where Squeeze Theorem is applied in the last step.

Let $x_n := \frac{1}{\sqrt{2n\pi}} \in [-1, 1]$ for $n \in \mathbb{N}$. Then

$$g'(x_n) = -2\sqrt{2n\pi} \to -\infty \qquad \text{as } n \to \infty.$$

Hence g' is not bounded on the interval $[-1, 1]$. □

6.2-7 Use the Mean Value Theorem to prove that $(x - 1)/x < \ln x < x - 1$ for $x > 1$. [Hint: Use that fact that $D \ln x = 1/x$ for $x > 0$.]

Solution. Let $f(t) = \ln t$. Fix $x > 1$. Then f is continuous on $[1, x]$ and differentiable on $(1, x)$ with $f'(t) = 1/t$. By Mean Value Theorem, there exists $c \in (1, x)$ such that

$$\frac{f(x) - f(1)}{x - 1} = f'(c),$$

that is,

$$\ln x = \frac{x-1}{c}.$$

Since $1 < c < x$, it follows that $1/x < 1/c < 1$, and hence

$$\frac{x-1}{x} < \ln x < x - 1.$$

□