MATH4210: Financial Mathematics Tutorial 4

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Review on Normal r.v.

Question

Assume a sequence of i.i.d. r.v.s $\{X_i\}_{i=1\dots n}$, $X_1 \sim N(\mu, \sigma^2)$. *Denote by* $Y := \frac{1}{n}\sum_{i=1}^{n} X_i$. *Find* $\mathbb{E}[Y]$ *and* $Var(Y)$

$$E[Y] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu.$$

$$var(Y) = var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} var\left(\sum_{i=1}^{n} X_i\right) + \sum_{i\neq j}\frac{1}{n^2}\underbrace{cov(X_i, X_j)}_{=0}$$

$$= \frac{1}{n^2} n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

Review on Normal r.v.

if $Y \sim N(\mu_Y, \sigma_Y) \Leftrightarrow E[e^{i\theta Y}] = e^{i\theta\mu_Y - \frac{1}{2}\theta^2\sigma_Y^2}$

Fix $\theta \in \mathbb{R}$

$$E[e^{i\theta Y}] = E[e^{i\theta\sum_{j=1}^{n} X_j}] = E[\prod_{j=1}^{n} e^{i\theta X_j}]$$

Question

Assume a sequence of independent r.v.s $\{X_i\}_{i=1\ldots n}$, for any $i \in [|1, n|]$, $X_i \sim N(\mu_i, \sigma_i^2)$. Denote by $Y := \sum_{i=1}^{n} X_i$. Show that Y is gaussian. Find the parameters of Y.

Since X_j's are independent.

$$E[e^{i\theta Y}] = \prod_{j=1}^{n} E[e^{i\theta X_j}] = \prod_{j=1}^{n} e^{i\theta\mu_j - \frac{1}{2}\theta^2\sigma_j^2}$$

$$= e^{i\theta\underbrace{\sum_{j=1}^{n}\mu_j}_{\mu_Y} - \frac{1}{2}\theta^2\underbrace{\sum_{j=1}^{n}\sigma_j^2}_{\sigma_Y^2}}$$

$$Y \sim N\left(\sum_{j=1}^{n}\mu_j, \sum_{j=1}^{n}\sigma_j^2\right)$$

$$f_X(x) = \frac{d}{dx} P_X(X \le x) = \frac{d}{dx} P_X(\ln X \le \ln x)$$

$\ln X \sim N(\mu, \sigma^2)$

Φ : cdf of $N(0,1)$

φ : pdf of $N(0,1)$

$$= \frac{d}{dx} \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

$$= \varphi\left(\frac{\ln x - \mu}{\sigma}\right) \frac{d}{dx}\left(\frac{\ln x - \mu}{\sigma}\right)$$

Question

Assume X follows log-normal distribution with parameters μ, σ^2. Find the probability density function of X.

Recall that $X \sim LN(\mu, \sigma^2)$ if

$$\ln(X) \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$= \varphi\left(\frac{\ln x - \mu}{\sigma}\right) \cdot \frac{1}{\sigma x}$$

$$= \frac{1}{x\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

fix $n \in \mathbb{N}^*$,

$$E[X^n] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n e^{-\frac{x^2}{2}} dx = \underline{\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{n-1} \cdot x \cdot e^{-\frac{x^2}{2}} dx}$$

$V(x) = e^{-\frac{x^2}{2}}$ $\quad u(x) = x^{n-1}$ $\quad = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} - u(x)\, dV(x)$

Question

Assume $X \sim N(0,1)$. For $n \in \mathbb{N}^*$, find $\mathbb{E}[X^n]$.

If n is odd $\quad E[X^n] = 0$

If n is even. $\quad E[X^n] = (n-1)(n-3)\cdots 1$

Remark

If $Y \sim N(\mu, \sigma^2)$, we can write $\mathbb{E}[Y^n]$ explicitly by considering

IDP

$$Y = \mu + \sigma Z, \quad Z \sim N(0,1).$$

$$E[X^n] = \begin{cases} 0 & \text{if } n = 2k-1 \\ \frac{(2k)!}{2^k \cdot k!} & \text{if } n = 2k \end{cases}$$

$$E[X^n] = \frac{1}{\sqrt{2\pi}} \left(-x^n e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} \right) - \left(\int_{\mathbb{R}} - e^{-\frac{x^2}{2}} (n-1) \cdot x^{n-2} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + n-1 \int_{\mathbb{R}} x^{n-2} e^{-\frac{x^2}{2}} dx \right)$$

$$= (n-1) E[X^{n-2}]$$

$E[X] = 0$

$E[X^2] = 1$

Convergence of r.v.s

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

$$\mathbb{P}[\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1$$

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

$$\lim_{n\to\infty} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

Convergence of r.v.s

since $X_n \to X$ a.s $\Rightarrow P\{w \in \Omega : \lim_{n\to\infty} X_n(w) = X(w)\} = 1$

① fix $\rho > 0$, and $n \in \mathbb{N}$

$A_n = \bigcup_{m \geq n} \{ |X_m - X| \geq \rho \}$

$\{|X_n - X| \geq \rho\} \subseteq A_n$

$\Rightarrow \lim P(\{|X_n - X| \geq \rho\}) \leq \lim P(A_n)$ (*)

② A_n is an decreasing sequence

$P(A_1) \leq P(\Omega) = 1$

$P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$ (**)

$=0$

Question

Let X and $\{X_n\}$ be $\mathbb{R}$ valued (sequence of) r.v.s. Assume $X_n \to X$ a.s.. Show that $X_n \to X$ in probability.

③ consider $B = \{ |\lim_{n\to\infty} X_n - X| = 0 \}$

$P(B) = 1$

Fix $w \in B$ since $X_n \to X$ a.s.

There exist $N > 0$

$|X_n(w) - X(w)| < \varepsilon$ for all $n \geq N$

But $\forall n \geq N$ $w \notin A_n \Rightarrow B \cap (\bigcap_{n=1}^{\infty} A_n) = \emptyset$

$P(B \cup (\bigcap_{n=1}^{\infty} A_n)) \leq P(\Omega)$

$= 1$

$= P(B) + P(\bigcap_{n=1}^{\infty} A_n) \leq 1$

$\Rightarrow P(\bigcap_{n=1}^{\infty} A_n) = 0$ (**)

$\Rightarrow X_n \to X$ in probability