MATH4210: Financial Mathematics Tutorial 2

YI SHEN

The Chinese University of Hong Kong

yishen@math.cuhk.edu.hk

17 September, 2024

Exercise

Suppose $X_k \sim N\left(\mu_k, \sigma_k^2\right)$, μ_k, σ_k convergent, and $X_k \to X$ in $\mathbb{L}^2$. Show X is a normal random variable with $\mathbb{E}[X] = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

Solution:
We claim that $X_k \to X$ in $\mathbb{L}^2$ implies $X_k \to X$ in $\mathbb{L}^1$. We accept this for now.
Fix $t \in \mathbb{R}$ and $k \in \mathbb{N}$, consider the characteristic function:

$$\mathbb{E}\left[|e^{itX_k} - e^{itX}|^2\right] \leq t^2 \mathbb{E}\left[|X_k - X|^2\right] \to 0 \text{ as } k \to \infty.$$

It follows from the fact: $\forall x, y \in \mathbb{R}, a \in \mathbb{R}$:

$$|e^{iay} - e^{iax}| = |\int_x^y iae^{ias} ds| \leq |a| \int_x^y 1 ds \leq |a||y - x|$$

Therefore, $e^{itX_k} \to e^{itX}$ in $\mathbb{L}^2$. By the claim, $e^{itX_k} \to e^{itX}$ in $\mathbb{L}^1$. Then

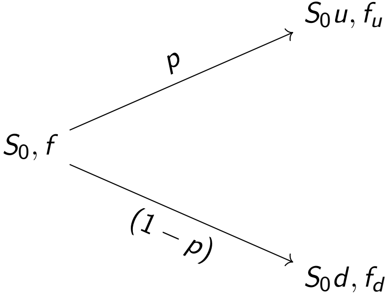
$$\mathbb{E}(e^{itX_k}) - \mathbb{E}(e^{itX}) \leq \mathbb{E}(|e^{itX_k} - e^{itX}|) \to 0$$

Hence, X is normal as characteristic function uniquely identifies distributions. Moreover, the characteristic functions of X coincides with limit of that of X_k's. By continuity, we then deduce $\mathbb{E}(X) = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

The claim can be proven by Jensen's inequality or Cauchy-Schwarz inequality.

$$\begin{aligned}\mathbb{E}(|X_k - X|) &= \mathbb{E}(\sqrt{|X_k - X|^2}) \\ &\leq \sqrt{\mathbb{E}(|X_k - X|^2)} \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Consider the following problem.
Suppose we target on a stock whose price is $S_t, t = 0, 1$. t represents the time. Now we are at time $t = 0$, and we can observe the stock price S_0. At time $t = 1$, it has two possibilities of moving to either $S_1 = S_0 u$ or $S_1 = S_0 d$ for some $u > 1$ and $0 < d < 1$ with probability p and $1 - p$ respectively. Suppose we have an option whose underlying asset is the stock maturing at $t = 1$ with strike K. How can we find the option price f at time $t = 0$?



Why do we want to compute the option price at time $t = 0$?
Suppose you are a product manager of some securities company. A customer came to your company and ask for a large amount of product of this kind of options for hedging reason. You are asked to price this option such that there is no bad guys who can earn for sure from your product. How do we do that? You cannot foresee S_1 because you don't have Doraemon.
So the problem becomes TO PRICE THE OPTION SUCH THAT THERE IS NO ARBITRAGE OPPORTUNITIES.
Otherwise, if the option is not well priced, then we can find a replicating portfolio that replicates the true value of the option. And a profit can be realized from the spread of values for sure.

Binomial Tree Models

We consider two portfolios with initial wealth $x = f$.

1. Long ϕ_0 stock and put $x - \phi_0 S_0$ cash in the bank with discrete compound rate r
2. Long 1 options

Then, then values of the portfolios are

$$\Pi_1(0) = (x - \phi_0 S_0) + \phi_0 S_0$$

$$\Pi_1(T) = \begin{cases} (x - \phi_0 S_0)(1 + rT) + \phi_0 S_0 u & \text{if stock price move upwards} \\ (x - \phi_0 S_0)(1 + rT) + \phi_0 S_0 d & \text{if stock price move downwards} \end{cases}$$

and we have

$$\Pi_2(0) = f$$

$$\Pi_2(T) = \begin{cases} f_u & \text{if stock price move upwards} \\ f_d & \text{if stock price move downwards} \end{cases}$$

Binomial Tree Models

Can we choose ϕ_0 such that $\Pi_1(T) = \Pi_2(T)$ and $\Pi_1(0) = \Pi_2(0)$?
Yes, it suffices to solve

$$\begin{aligned} x &= f \\ (f - \phi_0 S_0)(1 + rT) + \phi_0 S_0 u &= f_u \\ (f - \phi_0 S_0)(1 + rT) + \phi_0 S_0 d &= f_d. \end{aligned}$$

Solve it, we have

$$\begin{aligned} \phi_0 &= \frac{f_u - f_d}{S_0(u - d)} \\ f = x &= (1 + rT)^{-1}(f_u - \phi_0 S_0(u - (1 + rT))) \\ &= (1 + rT)^{-1}(q f_u + (1 - q) f_d), \end{aligned}$$

where

$$q := \frac{(1 + rT) - d}{u - d}$$

.

If $q \in (0,1)$, it defines a probability measure $\mathbb{Q}$ (totally unrelated to p) that,

$$\begin{cases} \mathbb{Q}[S_T = S_0 u] & = \mathbb{Q}[f_T = f_u] = q \\ \mathbb{Q}[S_T = S_0 d] & = \mathbb{Q}[f_T = f_d] = 1 - q. \end{cases}$$

Then, we have

$$f = (1 + rT)^{-1}\mathbb{E}^{\mathbb{Q}}[f_T] := \sum_{f_i} f_i \times \mathbb{Q}[f_t = f_i]$$

Note that we do Not necessarily have

$$p := \mathbb{P}(S_T = S_0 u) = q.$$

If this is the case, we call it risk neutral world. $\mathbb{Q}$ is called Equivalent (Local) Martingale Measure (E(L)MM).

Remark

- we call r is a discrete compound rate if we only compound the interest on each time periods. For example, starting at $t = 0$, we compound the interest at $t = \Delta t, 2\Delta t, ...$ and the interest values $(1 + r\Delta t)^1, (1 + r\Delta t)^2, ...$
- we call r a continuous compound rate if we continuously compound the interest on all $t \geq 0$. At any $t \geq 0$, the interest values e^{rt}.
- We take the example in the previous slide, one can observe

$$\mathbb{E}^{\mathbb{Q}}[(1 + rT)^{-1} S_T | \sigma(S_0)] = S_0 \text{ or } \mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T | \sigma(S_0)] = S_0$$

which tells you the DISCOUNTED value of stock price is a "discrete martingale" under $\mathbb{Q}$ (rigorous justification is needed). Similarly, one can deduce the DISCOUNTED (whatever which method you compound) option price is a "discrete martingale" under $\mathbb{Q}$. This is the reason why $\mathbb{Q}$ is also called E(L)MM.

Question

Given the current price of the underlying stock, $S(0) = 20$. The stock price goes up and down by $u = 1.2$ and $d = 0.67$,respectively. The one period risk-free interest rate is 10%.

a) Price a one period European call option with exercise price $K = 20$. Consider the continuous compound case.

b) Price a two period European call option with exercise price $K = 20$ in the tree. Consider the discrete compound case.

c) Price a two period European put option with exercise price $K = 20$ in the tree. Consider the continuous compound case.

d) Consider (c), suppose the put option is American. What is its price?

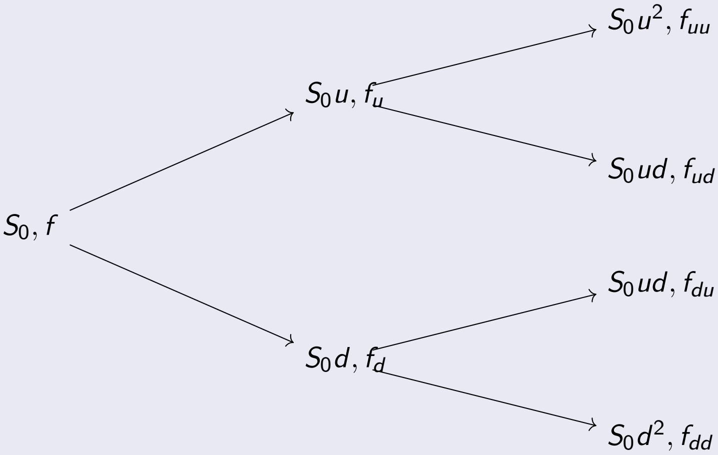
Answer

a)

1. $S(0) = 20,\ u = 1.2, d = 0.67, r = 0.1$
2. $S(0)u = 24,\ S(0)d = 13.4$
3. $q = \frac{e^{rT} - d}{u - d} \approx 0.82,\ 1 - q \approx 0.18$
4. $f_u = (S(0)u - K)^+ = 4,\ f_u = (S(0)d - K)^+ = 0$
5. $f = e^{-rT}(qf_u + (1-q)f_d) \approx e^{-0.1}(0.82 \times 4) = 2.9744$

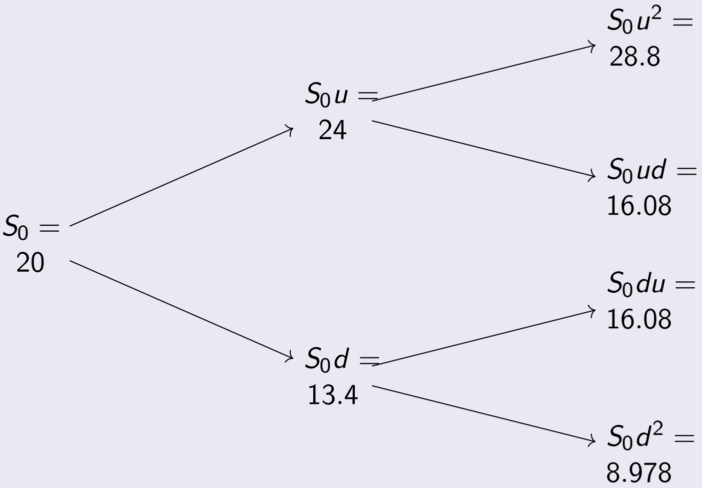
Binomial Tree Models

Answer



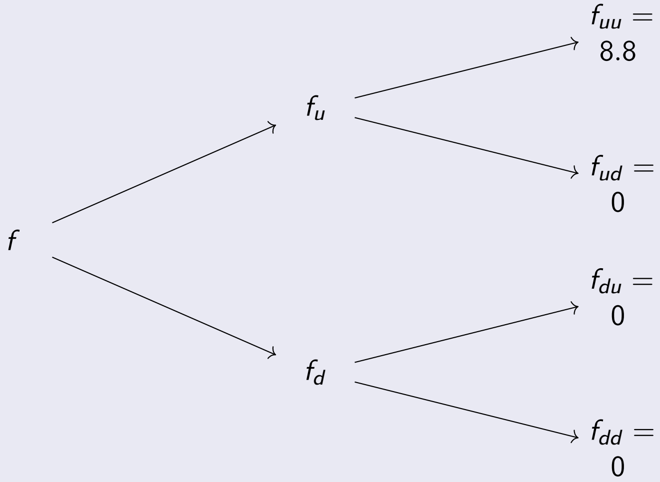
Binomial Tree Models

Answer



Binomial Tree Models

Answer



Binomial Tree Models

Answer

Compared with question (a), we use the discrete compound method, then in this case we should compute the EMM again:

$$q = \frac{(1 + r\Delta t) - d}{u - d} \approx 0.81, 1 - q \approx 0.19$$

According to the property of discounted price, if we are at the case $S_1 = uS_0$*:*

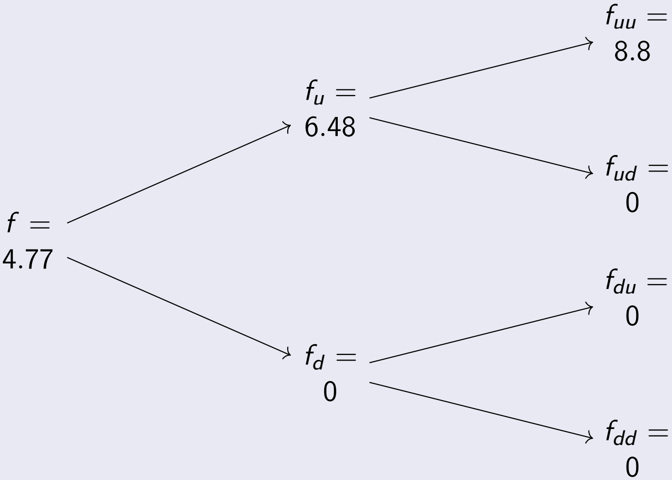
$$f_u = (1 + r\Delta t)^{-1}(q * f_{uu} + (1 - q) * f_{ud}) \approx 6.48$$

Remark: it can be written as $(1 + r\Delta t)^{-1} f_u = \mathbb{E}^{\mathbb{Q}}[(1 + r\Delta t)^{-2} f_2 | \sigma(S_0, S_1)]$.
Similarly, we compute $f_d = 0$. *Using* f_d *and* f_u*, we re-use the formula again to get* f*:*

$$f = (1 + r\Delta t)^{-1}(q * f_u + (1 - q) * f_d) \approx 4.77$$

Binomial Tree Models

Answer

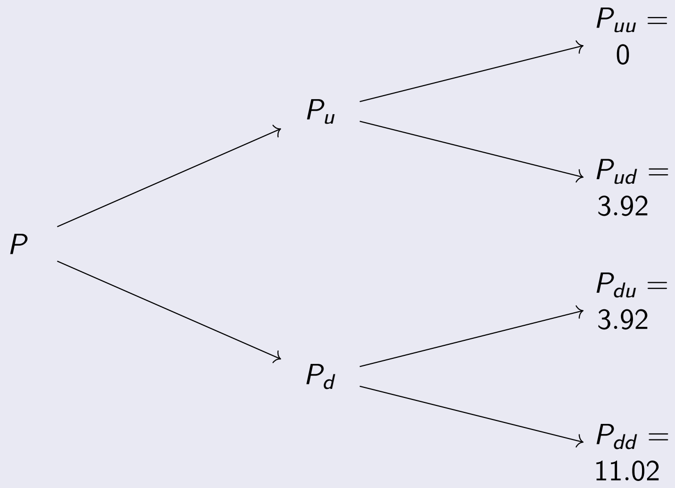


Binomial Tree Models

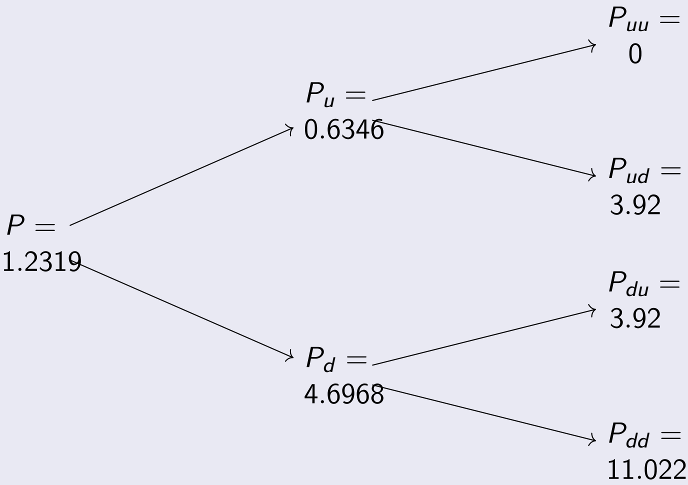
From the tree in the previous slide, one should be aware that we always do a backward computation. For those who are interested, it is good to redo question (b) in a continuous compound case. And you will notice that the solution are "almost" the same between these two cases. The reason is that the continuous compound mode is derived as a limit of discrete compound:

$$\lim_{n\to\infty}(1+\frac{r}{n})^{nt} = e^{rt}$$

Answer (c)



Answer



Binomial Tree Models

One can see that (similar result holds for discrete compound case)

Proposition

$f = e^{-2r\Delta t}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd})$ for a 2 period European option with period continuous compounding interest rate r.

One can also deduce from induction that

Proposition

$$f = e^{-nr\Delta t}\sum_{i=0}^{n} C_i^n q^i (1-q)^{n-i} f_{u^i d^{n-i}} = e^{-nr\Delta t}\mathbb{E}^{\mathbb{Q}}[f_{n\Delta t}]$$

for an n period European option with period continuous compounding interest rate r.

Question

What's the relation between European call and put with same settings?