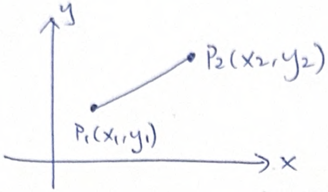
MATH 2010

Tutorial 1

Distance

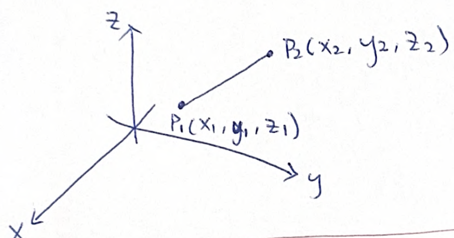
In 2-dimensional xy-plane:



Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

In 3-dimensional space ($\mathbb{R}^3$):



Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Example: The distance between $P_1(2,1,5)$ and $P_2(-2,3,0)$ is

$$|P_1P_2| = \sqrt{(-2-2)^2+(3-1)^2+(0-5)^2}$$

$$= \sqrt{16+4+25}$$

$$= \sqrt{45} \approx 6.708$$

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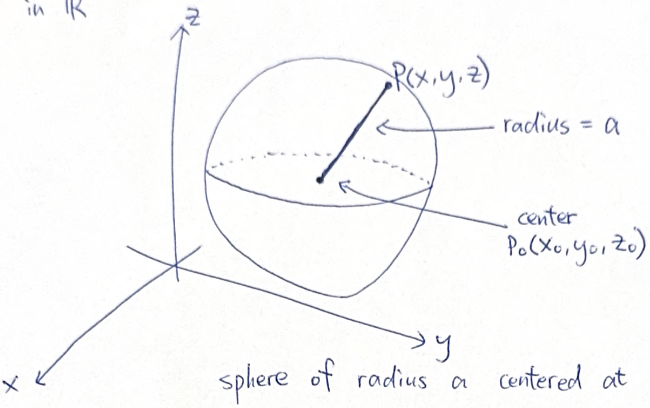
Sphere in $\mathbb{R}^3$



sphere of radius a centered at $P_0(x_0, y_0, z_0)$

For any point $P(x, y, z)$ on the sphere,

Distance between $P(x, y, z)$ and center $P_0(x_0, y_0, z_0)$ $= a$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = a$$

$$\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2}$$

The standard Equation for the sphere of radius a and center (x_0, y_0, z_0)

Example: Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

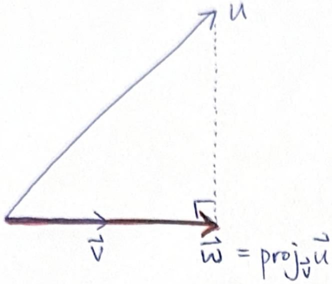
$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

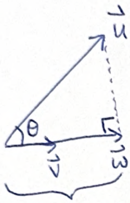
The center is $\left(-\frac{3}{2}, 0, 2\right)$ and the radius is $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

Projection



vector projection of $\vec{u}$ onto a nonzero vector $\vec{v}$

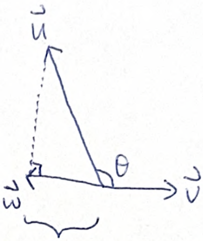
Notation : $\text{proj}_{\vec{v}}\vec{u}$



θ : angle between $\vec{u}$ and $\vec{v}$

length $= \|\vec{u}\| \cos\theta$

direction $= \dfrac{\vec{v}}{\|\vec{v}\|}$



length $= -\|\vec{u}\| \cos\theta$

direction $= -\dfrac{\vec{v}}{\|\vec{v}\|}$

$$\vec{w} = \underbrace{(\|\vec{u}\| \cos\theta)}_{\text{length}} \underbrace{\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}_{\substack{\text{direction} \\ \text{(unit vector)}}}$$

$$= \cancel{\|\vec{u}\|} \underbrace{\frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\|\vec{v}\|}}_{\cos\theta} \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

The number $\|\vec{u}\| \cos\theta$ (could be positive or negative) is called the <u>scalar component</u> of $\vec{u}$ in the direction of $\vec{v}$.

The vector projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\left(\frac{\vec{v}}{\|\vec{v}\|}\right).$$

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is

$$\|\vec{u}\|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|}.$$

Example: Find the vector projection of $\vec{u} = 6\vec{i}+3\vec{j}+2\vec{k}$ onto $\vec{v} = \vec{i}-2\vec{j}-2\vec{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{6-6-4}{1+4+4}(\vec{i}-2\vec{j}-2\vec{k})$$

$$= -\frac{4}{9}(\vec{i}-2\vec{j}-2\vec{k}) = -\frac{4}{9}\vec{i}+\frac{8}{9}\vec{j}+\frac{8}{9}\vec{k}.$$

Scalar component of $\vec{u}$ in the direction of $\vec{v}$

$$= \|\vec{u}\|\cos\theta = \vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|} = (6\vec{i}+3\vec{j}+2\vec{k})\cdot\left(\frac{1}{3}\vec{i}-\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)$$

$$= 2-2-\frac{4}{3} = -\frac{4}{3}$$

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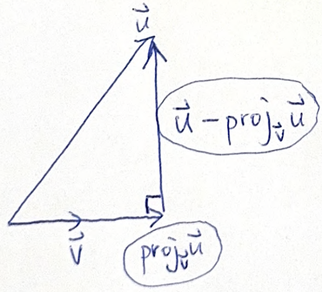
Example: Verify that the vector $\vec{u}-\text{proj}_{\vec{v}}\vec{u}$ is orthogonal to the projection vector $\text{proj}_{\vec{v}}\vec{u}$.



$\because$ The vector $\mathrm{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$ is parallel to $\vec{v}$.

$\therefore$ It suffices to show that the vector $\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$.

$$
\begin{aligned}
&(\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}) \cdot \vec{v} \\
&= \left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}\right)\cdot\vec{v} \\
&= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}\right)\cdot\vec{v} \\
&= \vec{u}\cdot\vec{v} - \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}}_{\text{scalar}}(\vec{v}\cdot\vec{v}) \\
&= \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{v}\|^2}}\cancel{\|\vec{v}\|^2} \\
&= 0
\end{aligned}
$$

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The vector $\vec{u}$ is expressed as a sum of two orthogonal vectors.

$$
\begin{aligned}
\vec{u} &= \mathrm{proj}_{\vec{v}}\vec{u} + (\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}) \\
&= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\vec{v} + \left(\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}\right)
\end{aligned}
$$