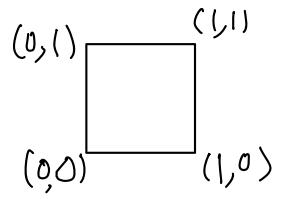


eg2 Find global max & min and values of

$$f(x,y) = 6xy - 4x^3 - 3y^2$$

on the set  $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$



Solu Step1 : Interior extremum

( $f$  is a polynomial,  $\vec{\nabla}f$  always exist)

$$\begin{cases} 0 = \frac{\partial f}{\partial x} = 6y - 12x^2 \\ 0 = \frac{\partial f}{\partial y} = 6x - 6y \end{cases}$$

$$\Rightarrow y = 2x^2 \quad \& \quad x = y$$

$$\Rightarrow x = 2x^2$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{1}{2}$$

Hence  $(x,y) = (0,0)$  &  $(x,y) = (\frac{1}{2}, \frac{1}{2})$

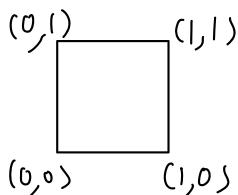
↑  
on the boundary

↑  
in the interior  $\rightarrow$  critical point.

with  $f(0,0) = 0$  &  $f(\frac{1}{2}, \frac{1}{2}) = \dots = \frac{1}{4}$  check

On

Boundary



$$(1) \quad \{x=0, 0 \leq y \leq 1\} : \quad f(0,y) = -3y^2$$

$$\Rightarrow f(0,1) = -3 \leq f(0,y) \leq 0 = f(0,0)$$

$$(2) \quad \{y=0, 0 \leq x \leq 1\} : \quad f(x, 0) = -4x^3$$

$$f(1, 0) = -4 \leq f(x, 0) \leq 0 = f(0, 0)$$

$$(3) \quad \{x=1, 0 \leq y \leq 1\} : \quad f(1, y) = 6y - 4 - 3y^2 \\ = -3(y-1)^2 - 1$$

$$f(1, 0) = -4 \leq f(1, y) \leq -1 = f(1, 1)$$

$$(4) \quad \{y=1, 0 \leq x \leq 1\} : \quad f(x, 1) = 6x - 4x^3 - 3$$

$$0 = \frac{d}{dx} f(x, 1) = 6 - 12x^2 \Rightarrow x = \frac{1}{\sqrt{2}}$$

$(-\frac{1}{\sqrt{2}}$  rejected because  $0 \leq x \leq 1$ )

$$f\left(\frac{1}{\sqrt{2}}, 1\right) = \dots = -(3 - 2\sqrt{2}) < 0$$

$$f(0, 1) = -3, \quad f(1, 1) = -1 (< -(3 - 2\sqrt{2}))$$

$$\Rightarrow f(0, 1) = -3 \leq f(x, 1) \leq -(3 - 2\sqrt{2}) = f\left(\frac{1}{\sqrt{2}}, 1\right)$$

Compare values, global max/min on boundary are  
 max:  $f(0, 0) = 0$ , min:  $f(1, 0) = -4$

Compare values

Global max. at  $(\frac{1}{2}, \frac{1}{2})$  with value  $\frac{1}{4}$

Global min. at  $(1, 0)$  with value  $-4$

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## Matrix form for 2<sup>nd</sup> order Taylor Polynomial

Def: Let  $f: \Omega \rightarrow \mathbb{R}$  be  $C^2$  ( $\Omega \subseteq \mathbb{R}^n$ , open).

Then the Hessian matrix of  $f$  at  $\vec{a} \in \Omega$  is

$$Hf(\vec{a}) = \begin{bmatrix} f_{x_1 x_1}(\vec{a}) & \cdots & f_{x_1 x_n}(\vec{a}) \\ \vdots & & \vdots \\ f_{x_n x_1}(\vec{a}) & \cdots & f_{x_n x_n}(\vec{a}) \end{bmatrix}$$

Remarks (1)  $Hf(\vec{a})$  is  $n \times n$  symmetric (by Clairaut's Thm)

(2) In Textbook, Hessian of  $f = \det(Hf(\vec{a}))$

So we emphasize our definition is a  
matrix (More common in advanced level math)

Eg:  $f(x,y)$  at  $(0,0)$

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \quad (f_{xy} = f_{yx})$$

$$\begin{bmatrix} x, y \end{bmatrix} \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2}_{\text{2nd order term in Taylor polynomials}}$$

(up to a factor  $\frac{1}{2!}$ )

2<sup>nd</sup> order Taylor polynomial of f at  $\vec{a}$  in matrix form

$$P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla}f(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2} (\vec{x}-\vec{a})^T Hf(\vec{a}) (\vec{x}-\vec{a})$$

Where  $\vec{\nabla}f(\vec{a})$  regarded as row vector  $[f_{x_1}(\vec{a}) \dots f_{x_n}(\vec{a})]$ ,

$\vec{x}-\vec{a}$  regarded as column vector  $\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{bmatrix}$

&  $(\vec{x}-\vec{a})^T$  is the transpose  $[x_1-a_1 \dots x_n-a_n]$   
(row vector)

Q  $g(x,y) = \frac{\ln x}{1-y}$ . Find  $P_2(x,y)$  at  $(1,0)$  using matrix form.

Soh :  $g(1,0) = 0$

$$\vec{\nabla}g = \left[ \frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \right] = \left[ \frac{1}{x(1-y)} \quad \frac{\ln x}{(1-y)^2} \right]$$

$$Hg = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial y \partial x} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \left( = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2\ln x}{(1-y)^3} \end{bmatrix}$$

$$\Rightarrow \vec{\nabla}g(1,0) = [1 \quad 0]$$

$$Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned}\therefore P_2(x,y) &= g(1,0) + \vec{\nabla}g(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix} \\ &\quad + \frac{1}{2} [x-1 \ y] Hg(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix} \\ &= 0 + [1 \ 0] \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \ y] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix} \\ &= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y \quad (\text{check!})\end{aligned}$$

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