eg1: $\vec{f}: \mathbb{R} \to \mathbb{R}^2$,

$$\vec{f}(\theta) = (\cos\theta, \sin\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\vec{g}: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\vec{g}(u,v) = \begin{bmatrix} 2uv \\ u^2 - v^2 \end{bmatrix}$$

Find $D(\vec{g} \circ \vec{f})(\theta)$. $\left(\begin{array}{l} u = \cos\theta \\ v = \sin\theta \end{array} \right)$

Soln: Method 1: Find composition explicitly

$$\vec{g} \circ \vec{f}(\theta) = \vec{g}(\cos\theta, \sin\theta) = \begin{bmatrix} 2\cos\theta\sin\theta \\ \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ \cos 2\theta \end{bmatrix}$$

$$D(\vec{g} \circ \vec{f})(\theta) = \begin{bmatrix} \frac{d}{d\theta} \sin 2\theta \\ \frac{d}{d\theta} \cos 2\theta \end{bmatrix} = \begin{bmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{bmatrix}$$

Method 2 Chain Rule

$$D\vec{f}(\theta) = \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} \frac{d}{d\theta}\cos\theta \\ \frac{d}{d\theta}\sin\theta \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$D\vec{g}(u,v) = \begin{bmatrix} -\vec{\nabla} g_1 - \\ -\vec{\nabla} g_2 - \end{bmatrix} = \begin{bmatrix} -\vec{\nabla}(2uv) - \\ -\vec{\nabla}(u^2 - v^2) - \end{bmatrix} = \begin{bmatrix} 2v & 2u \\ 2u & -2v \end{bmatrix}$$

By Chain rule

$$D(\vec{g} \circ \vec{f})(\theta) = D\vec{g}(\vec{f}(\theta)) D\vec{f}(\theta)$$

$$= \begin{bmatrix} 2\sin\theta & 2\cos\theta \\ 2\cos\theta & -2\sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} -2\sin^2\theta + 2\cos^2\theta \\ -2\cos\theta\sin\theta - 2\sin\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos 2\theta \\ -2\sin 2\theta \end{bmatrix}$$

\#

eg2 (Abuse of notations)

$$f(x,y) = (x^2, 3xy, x+y^2) \quad (= \vec{f})$$

$$g(u,v,w) = \frac{uw}{v}$$

Consider $g \circ f$: $\begin{matrix} x \\ y \end{matrix} \xrightarrow{f} \begin{matrix} (f_1 =)\ u \\ (f_2 =)\ v \\ (f_3 =)\ w \end{matrix} \xrightarrow{g} g$

Find $\frac{\partial g}{\partial x}(1,1)$ (regard g as a function of x, y)

Soln: (g is real valued, $Dg = \vec{\nabla} g$)

$$Dg = \vec{\nabla} g = \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w} \right] = \left[\frac{w}{v} \quad -\frac{uw}{v^2} \quad \frac{u}{v} \right]$$

At $(1,1)$, $\begin{cases} u = x^2 = 1 \\ v = 3xy = 3 \\ w = x + y^2 = 2 \end{cases}$ (i.e. $f(1,1) = (1,3,2)$)

$$Dg(f(1,1)) = Dg(1,3,2) = \left[\frac{2}{3} \quad -\frac{2}{9} \quad \frac{1}{3} \right]$$

$$D\vec{f} = \begin{bmatrix} \vec{\nabla} f_1 \\ \vec{\nabla} f_2 \\ \vec{\nabla} f_3 \end{bmatrix} = \begin{bmatrix} \vec{\nabla} x^2 \\ \vec{\nabla}(3xy) \\ \vec{\nabla}(x+y^2) \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 3y & 3x \\ 1 & 2y \end{bmatrix}$$

$$D\vec{f}(1,1) = \begin{bmatrix} 2 & 0 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

Hence Chain rule $\Rightarrow$ $D(g \circ f)(1,1) = \left[\frac{2}{3} \quad -\frac{2}{9} \quad \frac{1}{3} \right] \begin{bmatrix} 2 & 0 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$

$= [1 \quad 0]$ (Check!)

$\leftarrow \left[\frac{\partial g}{\partial x}(1,1) \quad \frac{\partial g}{\partial y}(1,1)\right]$

$\Rightarrow \quad \frac{\partial g}{\partial x}(1,1) = 1$ ※

Remark: We should just calculate the 1st column

$$\underbrace{\left[\frac{2}{3}, -\frac{2}{9}, \frac{1}{3}\right]}_{\left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial g}{\partial w}\right]} \begin{bmatrix} 2 & * \\ 3 & * \\ 1 & * \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & * \\ \frac{\partial f_2}{\partial x} & * \\ \frac{\partial f_3}{\partial x} & * \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} \qquad \begin{pmatrix} u = f_1 \\ v = f_2 \\ w = f_3 \end{pmatrix}$$

In general, for 2-variables $\xrightarrow{\vec{f}}$ 3-variables $\xrightarrow{g}$ real-valued

i.e. $k = 2$, $n = 3$, $m = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\vec{f}} \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{pmatrix} \xrightarrow{g} g(f_1, f_2, f_3)$$

we usually use classical notation,

(x,y) for (x_1, x_2),

$u = f_1(x,y)$, $v = f_2(x,y)$, $w = f_3(x,y)$, and

$g = g(u,v,w)$

$(x,y) \longmapsto (u,v,w) \longmapsto g$ can be regarded as function of (x,y) : $g = g(x,y)$

$\uparrow$ Abuse of notations

Then the Chain rule is

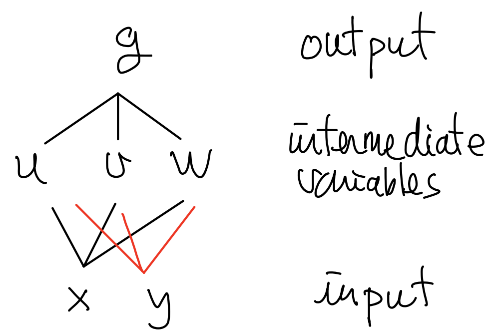
$$\boxed{\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial x} \\ \frac{\partial g}{\partial y} &= \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial g}{\partial w}\cdot\frac{\partial w}{\partial y} \end{aligned}}$$

$$\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = D(g\circ\vec{f}) = Dg(\vec{f})\, D\vec{f} = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}$$

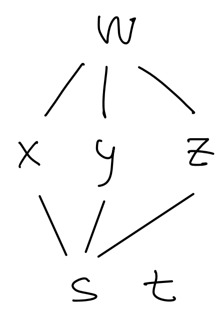
(1×2 matrix) (1×3)(3×2) matrix

(Similarly for other low dimensional situations)

<u>Remark</u>: Branch Diagram
(in Textbook)



eg 3 $\quad w(x,y,z) = \sqrt{x^2+y^2+z^2}$



$$\begin{cases} x = 3e^{t}\sin s \\ y = 3e^{t}\cos s \\ z = 4e^{t} \end{cases}$$

Find $\dfrac{\partial w}{\partial s}$ at $s=t=0$.

Soln:
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\right)\left(\frac{\partial}{\partial s}(3e^{t}\sin s)\right)$$
$$+ \left(\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}\right)\left(\frac{\partial}{\partial s}(3e^{t}\cos s)\right)$$
$$\left(\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}\right)\left(\frac{\partial}{\partial s}(4e^{t})\right)$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot 3e^{t}\cos s + \frac{y}{\sqrt{x^2+y^2+z^2}}\cdot(-3e^{t}\sin s) + 0$$

Put $s=t=0 \Rightarrow \quad x=0,\ y=3,\ z=4$

$$\frac{\partial w}{\partial s}(0,0) = 0+0 = 0$$

※

eg4. John is walking with position at time t given by

$$\begin{cases} x(t) = t^3+1 \\ y(t) = 2t^2 \end{cases}$$

Altitude is $H(x,y) = x^2 - y^2 + 100$

(1) Is John going up/down at $t=1$?

(2) Which direction should he go instead at $t=1$ to go down most quickly ?

Soln: (1) Find $\left.\dfrac{dH}{dt}\right|_{t=1}$

$$\begin{tikzcd}[row sep=small] & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & H \\ & y \arrow[ur] & \end{tikzcd}$$

Chain rule $\quad \dfrac{dH}{dt} = \dfrac{\partial H}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial H}{\partial y}\dfrac{dy}{dt}$

$$= 2x \cdot 3t^2 - 2y \cdot 4t$$

At $t=1$, $(x,y) = (2,2)$

$\therefore \quad \left.\dfrac{dH}{dt}\right|_{t=1} = 2 \cdot 2 \cdot 3 1^2 - 2 \cdot 2 \cdot 4 \cdot 1 = -4 < 0$

$\therefore$ John is going <u>down</u> at $t=1$

(2) Go down most quickly in the direction of $-\vec{\nabla} H$

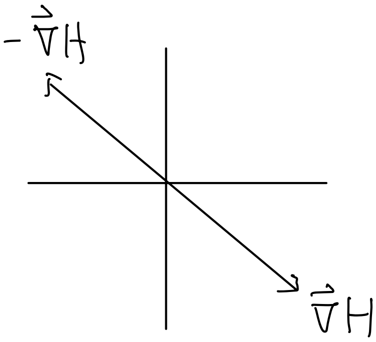
(by the geometric interpretation of $\vec{\nabla}$)

$\vec{\nabla} H = (2x, -2y) \qquad$ at $(x,y) = (2,2)$ (when $t=1$)

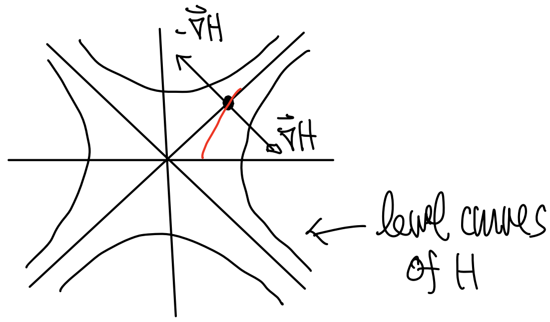
$= (4, -4)$

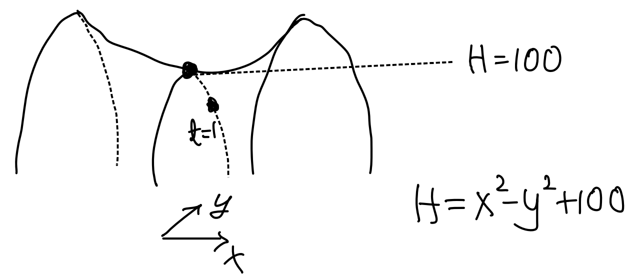
$\therefore$ H decreases most rapidly in the direction of

$$-\vec\nabla H(2,2) = (-4,4)$$



i.e John should go NW (Northwest)





<u>Idea of Pf of Chain Rule</u>

- $\vec f : \Omega_1 \to \mathbb{R}^n$ ($\Omega_1 \subseteq \mathbb{R}^k$, open)
- $\vec g : \Omega_2 \to \mathbb{R}^m$ ($\Omega_2 \subseteq \mathbb{R}^n$, open)
- $\vec f(\Omega_1) \subset \Omega_2$,
- $\vec f$ differentiable at $\vec a \in \Omega_1 \subset \mathbb{R}^k$
- $\vec g$ differentiable at $\vec b = \vec f(\vec a) \in \Omega_2 \subset \mathbb{R}^n$

$$\begin{array}{ccccc} \Omega_1 & \xrightarrow{\vec f} & \Omega_2 & \xrightarrow{\vec g} & \mathbb{R}^m \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ \vec x & & \vec y & & \\ & & \| & & \\ & & \vec f(\vec x) & & \end{array}$$

$$\begin{cases} \vec f(\vec x) - \vec f(\vec a) = D\vec f(\vec a)(\vec x - \vec a) + \vec\varepsilon_{\vec f}(\vec x) & \text{—— (1)} \\ \vec g(\vec y) - \vec g(\vec b) = D\vec g(\vec b)(\vec y - \vec b) + \vec\varepsilon_{\vec g}(\vec y) & \text{—— (2)} \end{cases}$$

Put $\vec{y} = \vec{f}(\vec{x})$ (and $\vec{b} = \vec{f}(\vec{a})$) into (2), we have

$$\vec{g}(\vec{f}(\vec{x})) - \vec{g}(\vec{b}) = D\vec{g}(\vec{b})(\vec{f}(\vec{x}) - \vec{f}(\vec{a})) + \vec{\varepsilon}_{\vec{g}}(\vec{f}(\vec{x}))$$

(by (1))

$$= D\vec{g}(\vec{b})\left[D\vec{f}(\vec{a})(\vec{x}-\vec{a}) + \vec{\varepsilon}_{\vec{f}}(\vec{x})\right] + \vec{\varepsilon}_{\vec{g}}(\vec{f}(\vec{x}))$$

$$= D\vec{g}(\vec{b})D\vec{f}(\vec{a})(\vec{x}-\vec{a})$$
$$+ \left[D\vec{g}(\vec{b})\vec{\varepsilon}_{\vec{f}}(\vec{x}) + \vec{\varepsilon}_{\vec{g}}(\vec{f}(\vec{x}))\right]$$

(expecting $\vec{\varepsilon}_{\vec{g}\circ\vec{f}}(\vec{x}) = D\vec{g}(\vec{b})\vec{\varepsilon}_{\vec{f}}(\vec{x}) + \vec{\varepsilon}_{\vec{g}}(\vec{f}(\vec{x}))$)

So we need to show that

$$\lim_{\vec{x}\to\vec{a}} \frac{\| D\vec{g}(\vec{b})\vec{\varepsilon}_{\vec{f}}(\vec{x}) + \vec{\varepsilon}_{\vec{g}}(\vec{f}(\vec{x}))\|}{\|\vec{x}-\vec{a}\|} = 0$$

(Proof Omitted
MATH2050 fa
1-variable case)

If we can do that, then we must have

$$D(\vec{g}\circ\vec{f})(\vec{x}) = D\vec{g}(\vec{b})D\vec{f}(\vec{a})$$

$$= D\vec{g}(\vec{f}(\vec{x}))D\vec{f}(\vec{a}).$$

\#

Summary: Jacobian Matrix

(1) 1-variable, real-valued : $f : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto f(x)$ ($x \in \Omega$, $f(x) \in \mathbb{R}$)

$$Df(x) = \frac{df}{dx} \qquad (1 \times 1 \text{ matrix, a scalar})$$

(2) multivariable, real-valued $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $\vec{x} \longmapsto f(\vec{x})$ ($\vec{x} \in \Omega$, $f(\vec{x}) \in \mathbb{R}$)

$$Df(\vec{x}) = \vec{\nabla} f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(\vec{x}), \cdots, \frac{\partial f}{\partial x_n}(\vec{x}) \right)$$

($1 \times n$ matrix, row-vector in $\mathbb{R}^n$)

(3) Multivariable, vector-valued $\vec{f} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\vec{x} \longmapsto \vec{f}(\vec{x})$ ($\vec{x} \in \Omega$, $\vec{f}(\vec{x}) \in \mathbb{R}^m$)

$$D\vec{f}(\vec{x}) = \begin{bmatrix} - \vec{\nabla} f_1 - \\ \vdots \\ - \vec{\nabla} f_m - \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad (m \times n \text{ matrix})$$

(4) 1-variable, vector-valued (a curve in $\mathbb{R}^m$) $\vec{\gamma} : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$, $t \longmapsto \vec{\gamma}(t) = (x_1(t), \cdots, x_m(t))$ ($t \in I$, $\vec{\gamma}(t) \in \mathbb{R}^m$)

$$D\vec{\gamma}(t) = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_m}{dt} \end{bmatrix} \qquad (m \times 1 \text{ matrix, column-vector in } \mathbb{R}^m)$$

Claim Rule in classical notation

$$(x_1, \cdots, x_k) \longrightarrow (y_1, \cdots, y_n) \longrightarrow (g_1, \cdots, g_m)$$

$$\begin{cases} g_i = g_i(y_1, \cdots, y_n) & \text{are functions of } y_1, \cdots, y_n \\ y_j = y_j(x_1, \cdots, x_k) & \text{are functions of } x_1, \cdots, x_k \end{cases}$$

We can regard $g_i = g_i(x_1, \cdots, x_k)$ as functions of $x_1, \cdots, x_k$

abuse of notation

Then

$$\underset{(m \times k)}{\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_k} \end{bmatrix}} = \underset{(m \times n)}{\begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_n} \end{bmatrix}} \underset{(n \times k)}{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_k} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_k} \end{bmatrix}}$$

is

$$\boxed{\begin{aligned} \frac{\partial g_i}{\partial x_j} &= \frac{\partial g_i}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial y_n}\frac{\partial y_n}{\partial x_j} \\ &= \sum_{l=1}^{n} \frac{\partial g_i}{\partial y_l}\frac{\partial y_l}{\partial x_j} \end{aligned}}$$

(for $i = 1, \cdots, m$, & $j = 1, \cdots, k$)