Properties of Limits

Assuming all limits on the right hand side exist, then the limit on the left hand side exists and the formula holds

(1) $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

(2) $\lim_{\vec{x}\to\vec{a}} k f(\vec{x}) = k \lim_{\vec{x}\to\vec{a}} f(\vec{x})$, where k is a constant

(3) $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \lim_{\vec{x}\to\vec{a}} g(\vec{x})$

(4) $\lim_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$

(5) $\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^n = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^n$, $n \geq 0$ (integer)

(6) $\lim_{\vec{x}\to\vec{a}} \left[f(\vec{x})\right]^{\frac{1}{n}} = \left[\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right]^{\frac{1}{n}}$, $n \geq 0$ (integer)
(If n is even, assume $f(\vec{x}) \geq$ near $\vec{a}$.)

Squeeze Theorem (Sandwich Theorem)

Let $f, g, h : \Omega \overset{\subset \mathbb{R}^n}{\longrightarrow} \mathbb{R}$ be functions of n-variables

If $\begin{cases} \bullet\ g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x}) \text{ near } \vec{a} \in \Omega \text{ and} \\ \bullet\ \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = \lim_{\vec{x} \to \vec{a}} h(\vec{x}) = L. \end{cases}$

Then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$

Special Case of Squeeze Theorem

If $\begin{cases} \bullet\ |f(\vec{x})| \leq g(\vec{x}) \text{ near } \vec{a} \text{ and} \\ \bullet\ \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = 0 \end{cases}$

Then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = 0$.

$\left(|f| \leq g \Rightarrow -g \leq f \leq g \right)$

eg: $\lim_{(x,y) \to (0,0)} x \cos\left(\frac{1}{x^2+y^2}\right)$

Soln: $\left| x \cos\left(\frac{1}{x^2+y^2}\right) \right| = |x| \left| \cos\left(\frac{1}{x^2+y^2}\right) \right| \leq |x|$

$\lim_{(x,y) \to (0,0)} |x| = 0$, Squeeze Thm $\Rightarrow \lim_{(x,y) \to (0,0)} x \cos\left(\frac{1}{x^2+y^2}\right) = 0$

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eg : $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$ ($\ln x$ = natural log = $\log x$)

Soln : $\left| \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} \right| = \frac{(x-1)^2}{(x-1)^2+y^2} |\ln x| \le |\ln x|$

$\lim_{(x,y)\to(1,0)} |\ln x| = 0$ hence Squeeze Thm $\Rightarrow$

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0 .$$

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Finding Limit using Polar Coordinates

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow$ " $(x,y)\to(0,0) \Leftrightarrow r\to 0$ "

eg Find limit $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ using polar coordinates.

Soln : Sub. $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2}$$

$$= \lim_{r\to 0} r(\cos^3\theta+\sin^3\theta) = 0 .$$

Since $|r(\cos^3\theta+\sin^3\theta)| \le r(|\cos^3\theta|+|\sin^3\theta|) \le 2r$

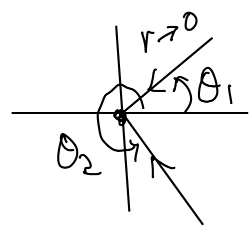
& $\lim_{r\to 0} r = 0$, Squeeze Thm implies the above limit. #

eg Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$ DNE.

Soln: $\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)} = \lim_{r\to 0} \frac{\cos^2\theta + \cos\theta\sin\theta}{2}$

Different θ gives different limits $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$ DNE

(Different θ means "approaching $(0,0)$ in different directions")



eg Find $\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

Soln $\lim_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = \lim_{r\to 0}\left(2r^2\ln r\right)\sin\theta\cos\theta = 0$

(by L'Hopital's Rule)

(& the squeeze thm with $|\sin\theta\cos\theta| \leq 1$)

Iterated Limit

(1) $\lim_{x\to a}\lim_{y\to b} f(x,y) \overset{\text{def}}{=\!=} \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$

i.e. 1st take limit as $y\to b$, then take limit as $x\to a$.

(2) Similarly for $\lim_{y\to b}\lim_{x\to a} f(x,y)$

(3) Are they equal? $\left(\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)\,?\right)$

(4) Relation to $\lim_{(x,y)\to(a,b)} f(x,y)$

eg: Consider $f(x,y) = \dfrac{x+y}{x-y}$

Soln

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x+y}{x-y} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} = \lim_{y\to 0}\frac{y}{-y} = \lim_{y\to 0}(-1) = -1$$

$$\therefore \quad \lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y)$$

Also $\lim_{(x,y)\to(0,0)}\dfrac{x+y}{x-y}$ DNE (Check! using "2 different paths have different limits")

Remarks

(eg $f(x,y)=\begin{cases}1, & \text{if } x=y\\ 0, & \text{if } x\neq y\end{cases}$, $(a,b)=(0,0)$) (Ex!)

(1)

$$\boxed{\begin{array}{ccc} \lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y) = \lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y) & \not\Longrightarrow & \lim\limits_{(x,y)\to(a,b)} f(x,y) \\ \text{both exist \& equal} & \not\Longleftarrow & \text{exists (\& equal)} \end{array}}$$

(eg, $f(x,y)=\begin{cases} x\cos\frac{1}{y}+y\cos\frac{1}{x}, & \text{if } x,y\neq 0\\ 0 & \text{if } x=0 \text{ or } y=0\end{cases}$ $(a,b)=(0,0)$) (Optional Ex!)

(2) If <u>all 3 limits exist</u>, then they are <u>equal</u>!

Continuity

Def: Let $f: A \xrightarrow{} \mathbb{R}$ ($A \subseteq \mathbb{R}^n$) & $\vec{a} \in A$ (so $f(\vec{a})$ is defined)

Then f is said to be <u>continuous at $\vec{a}$</u>

if $\boxed{\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})}$ (exists & equal to $f(\vec{a})$.)

Equivalently, $\forall \varepsilon > 0, \exists \delta > 0$ such that

if $\vec{x} \in A$ & $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Def $f: A \to \mathbb{R}$ is said to be <u>continuous</u> (on A) if f is <u>continuous at every point</u> in A.

<u>eg</u>: Let $k = 1, \cdots, n$. Show that $f(x_1, \cdots, x_n) = x_k$ is continuous on $\mathbb{R}^n$

(usually called the k-th coordinate function)

<u>Pf</u>: Let $\vec{a} = (a_1, \cdots, a_k, \cdots, a_n) \in \mathbb{R}^n$

$\forall \varepsilon > 0$, take $\delta = \varepsilon$

Then (for $\vec{x} \in \mathbb{R}^n$) $\|\vec{x} - \vec{a}\| < \delta$ implies

$$|f(\vec{x}) - f(\vec{a})| = |x_k - a_k| \leq \sqrt{(x_1-a_1)^2 + \cdots + (x_k - a_k)^2 + \cdots + (x_n - a_n)^2}$$

$$< \delta = \varepsilon$$

Hence $f(\vec{x}) = x_k$ is continuous at $\vec{a}$.

Since $\vec{a} \in \mathbb{R}^n$ is arbitrary, we've proved that f is cts. on $\mathbb{R}^n$ $\#$