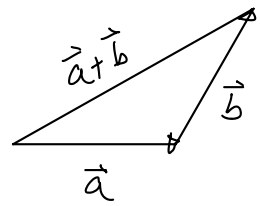
Triangle Inequality

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$



Equality holds $\Leftrightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \geq 0$

Pf: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$

$\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$ by Cauchy-Schwarz

$= (\|\vec{a}\| + \|\vec{b}\|)^2$

$\Rightarrow \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

For the equality part:

Equality holds $\Leftrightarrow$ $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|$ ——— (*)

$\Rightarrow$ Equality holds for Cauchy-Schwarz

$\Rightarrow$ $\vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$ for some $r \in \mathbb{R}$

Putting back into (*) $\left.\begin{array}{l} r\|\vec{b}\|^2 = \|\vec{a}\|\|\vec{b}\| \\ \text{or} \quad r\|\vec{a}\|^2 = \|\vec{a}\|\|\vec{b}\| \end{array}\right\} \Rightarrow r \geq 0.$ provided $\vec{a} \neq \vec{0}$ or $\vec{b} \neq \vec{0}$.

If $\vec{a} = \vec{b} = \vec{0}$, the statement is trivially correct ※

Option Ex: Cauchy-Schwarz inequality $\Leftrightarrow$ Triangle Inequality.
("$\Rightarrow$" done, "$\Leftarrow$" Ex.)

Special structure of $\mathbb{R}^3$: Cross Product $\vec{a}\times\vec{b}$

Let $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in\mathbb{R}^3$

Then the cross product $\vec{a}\times\vec{b}$ is defined by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

where $\hat{i}=(1,0,0)$, $\hat{j}=(0,1,0)$, $\hat{k}=(0,0,1)$.

eg: Let $\vec{a}=(2,3,5)$ & $\vec{b}=(1,2,3)$, then

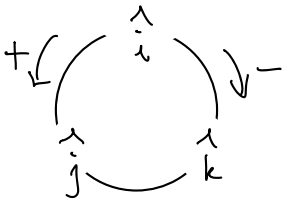
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\hat{k}$$

$$= -\hat{i} - \hat{j} + \hat{k} = (-1,-1,1)$$

Remark

(check!)

$\hat{i}\times\hat{i}=\vec{0}$	$\hat{i}\times\hat{j}=\hat{k}$	$\hat{i}\times\hat{k}=-\hat{j}$
$\hat{j}\times\hat{i}=-\hat{k}$	$\hat{j}\times\hat{j}=\vec{0}$	$\hat{j}\times\hat{k}=\hat{i}$
$\hat{k}\times\hat{i}=\hat{j}$	$\hat{k}\times\hat{j}=-\hat{i}$	$\hat{k}\times\hat{k}=\vec{0}$



Properties of Cross Product

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

<u>Algebraic</u> ((1) & (2) follow from properties of determinant)

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $(\alpha \vec{a} + \beta \vec{b}) \times \vec{c} = \alpha\, \vec{a} \times \vec{c} + \beta\, \vec{b} \times \vec{c}$

$\vec{a} \times (\alpha \vec{b} + \beta \vec{c}) = \alpha\, \vec{a} \times \vec{b} + \beta\, \vec{a} \times \vec{c}$

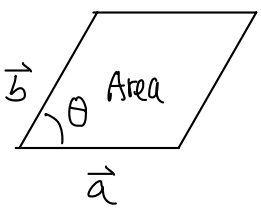
(3) $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ (easy from definition) (Ex)

<u>Geometric</u>

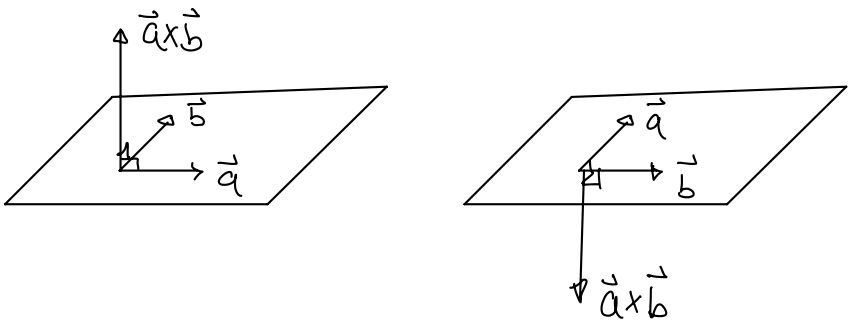
(4) Let θ = angle between $\vec{a}$ & $\vec{b}$, then

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

= Area of the parallelogram spanned by $\vec{a}$ & $\vec{b}$



<u>Remarks</u> (i) Formula (3) $\Rightarrow$ $\vec{a} \times \vec{b} \perp \vec{a}$ & $\vec{a} \times \vec{b} \perp \vec{b}$



(Also $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy <u>right-hand</u> rule (by checking definition))

(ii) Formula (4): $\vec{a}\times\vec{b}=\vec{0} \iff \text{Area}(\text{▱})=0$

$\iff \vec{a}=r\vec{b}$ or $\vec{b}=r\vec{a}$ for some $r\in\mathbb{R}$

$\iff \vec{a},\vec{b}$ are linearly <u>dependent</u>

(linear algebra)

<u>Pf</u> of (4):

By straight forward calculation (expanding both sides using def$^{\underline{n}}$)

$$\|\vec{a}\times\vec{b}\|^2 = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\right)^2 + \left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\right)^2 + \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)^2$$

$$= \cdots \text{ (Ex!)}$$

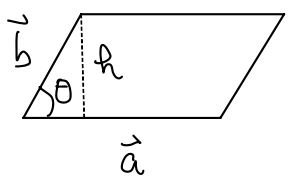
$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\|\vec{a}\|\|\vec{b}\|\cos\theta)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

$$\Rightarrow \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta \qquad (0\le\theta\le\pi)$$



$h=\|\vec{b}\|\sin\theta$

$$\Rightarrow \text{Area}(\text{▱}) = h\|\vec{a}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$$

$$= \|\vec{a}\times\vec{b}\|$$

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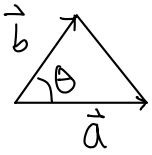
<u>Remarks</u>: (i) Area of 

$= \frac{1}{2}\|\vec{a}\times\vec{b}\|$

(ii) $\vec{a},\vec{b}\in\mathbb{R}^2$, i.e. $\vec{a}=(a_1,a_2,0)$ & $\vec{b}=(b_1,b_2 0)$

$$\vec{a} \times \vec{b} = \left(0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$$

$$\Rightarrow \quad \text{Area}(\text{▱}) = \left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$$ absolute value of determinant

$$= \left| \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right|$$