MATH 3030 ALGEBRA I

Lecture 1

Free groups (Motivation)

Let A be a set.

Consider the category $\mathcal{F}^A$ with

- $\mathrm{Obj}(\mathcal{F}^A) = \{(j, G) : G \text{ is a group and } j : A \to G \text{ is a map}\}$
- morphisms between (j_1, G_1) and (j_2, G_2) are comm. diagrams

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ & {\scriptstyle j_1}\nwarrow \quad \nearrow{\scriptstyle j_2} & \\ & A & \end{array}$$

s.t. φ is a group homomorphism

A free group $F(A)$ on A is then (the group component) of an initial object in $\mathcal{F}^A$. As such, it's unique up to isomorphism.

Free groups (Construction)

Given a set A, let A^{-1} be a set disjoint from A and such that there is a bijection:

$$A \xrightarrow{\sim} A^{-1}, \quad a \longleftrightarrow a^{-1}$$

A word on A is a finite ordered list

$$(a_1, a_2, \ldots, a_n)$$

which we denote by juxtaposition

$$w = a_1 a_2 \cdots a_n$$

where each "letter" a_i is either an elt $a \in A$ or an elt $a^{-1} \in A^{-1}$.
The set of words on A (including the empty word) is denoted by $W(A)$.
The number of letters in a word $w \in W(A)$ is called its length.

Note that there may be redundance in a word, e.g.

$aba^{-1}abb^{-1}bbaa^{-1}$ should be the same as $abbb$.

To resolve this, we introduce an elementary reduction map

$$r : W(A) \longrightarrow W(A)$$

$$w \longmapsto \text{removing the first occurred (from left to right) pair } aa^{-1} \text{ or } a^{-1}a \text{ from } w$$

e.g. $r(aba^{-1}abb^{-1}bbaa^{-1}) = abbb^{-1}bbaa^{-1}$

$r^2(aba^{-1}abb^{-1}bbaa^{-1}) = abbbaa^{-1}$

$r^3(aba^{-1}abb^{-1}bbaa^{-1}) = abbb$

We say that $w \in W(A)$ is a reduced word if $r(w) = w$.

Lemma If $w \in W(A)$ has length n, then $r^{\lfloor \frac{n}{2} \rfloor}(w)$ is reduced.

Now we can define the reduction map

$$R : W(A) \longrightarrow W(A)$$

$$w \longmapsto r^{\lfloor \frac{n}{2} \rfloor}(w) \text{ if } \text{length}(w) = n.$$

Let $F(A) = \text{Im}\, R$ = the set of all reduced words on A.

Define a binary operation on $F(A)$ by juxtaposition and reduction:

$$w \cdot w' := R(ww')$$

Thm $F(A)$ is a group under this operation.

Sketch of Pf:

- The operation is associative: this amounts to showing that reduction of words is independent of the order of iterating cancellations of pairs aa^{-1} or $a^{-1}a$.
- empty word is the identity
- inverse is given by reversing order of letters and replacing a by a^{-1} (or a^{-1} by a). #

e.g. $F(\emptyset) = \{e\}$ and if $A = \{a\}$ is a singleton, then $F(A) = \langle a \rangle \cong \mathbb{Z}$.
But if $|A| \geq 2$, $F(A)$ is quite complicated.

Rmk: We have a natural map $j : A \to F(A)$, $a \mapsto a$, and the pair $(j, F(A))$ is an initial object in $\mathcal{F}^A$.
So $F(A)$ gives the free group on A.

Rmk: Similarly, given a set A, we can define
the free abelian group on A

$$\mathbb{Z}^{\oplus A} := \{ f : A \to \mathbb{Z} \mid f(a) \neq 0 \text{ for only finitely many } a \in A \}$$

and the free vector space (over a field F) on A

$$F^{\oplus A} := \{ f : A \to F \mid f(a) \neq 0 \text{ for only finitely many } a \in A \}$$

Normal subgroups & Quotient groups

Given $H<G$, it's natural to ask if one can define a group structure on the set of left cosets $\{aH \mid a \in G\}$ by

$$(aH)\cdot(bH) := (ab)H \qquad (*)$$

Rmk: We can of course replace left cosets by right cosets in this discussion.

It turns out that this is not well-defined unless H satisfies a condition:

Def A subgroup $H < G$ is normal iff $aH = Ha$ $\forall a \in G$.
We denote a normal subgroup by $H \triangleleft G$.

Rmk: TFAE (why?)

(a) $H \trianglelefteq G$ i.e. $aH = Ha \ \forall a \in G$

(b) $aH \subset Ha \ \forall a \in G$

(c) $aHa^{-1} \subset H \ \forall a \in G$

(d) $aHa^{-1} = H \ \forall a \in G$

Thm Let $H < G$. Then the operation on left cosets

$$(aH)(bH) = (ab)H \qquad (*)$$

is well-defined iff $H \trianglelefteq G$.

Pf: $(*)$ is well-defined iff $aH = a'H$ & $bH = b'H \Rightarrow (ab)H = (a'b')H$
$\forall$ possible choices of a, a', b, b'.

iff $(ahbh')H = (ab)H \ \forall a, b \in G$ & $\forall h, h' \in H$.

($\Rightarrow$): Suppose that the operation $(*)$ is well-defined.

This means that

$$(ahbh')H = (ab)H \quad \forall h, h' \in H \text{ and } \forall a, b \in G.$$

In particular, $ahb \in abH \quad \forall h \in H$ and $\forall a, b \in G$

So $aha^{-1} \in H \quad \forall h \in H$ and $\forall a \in G$.

$\Rightarrow aHa^{-1} \subset H \quad \forall a \in G$. Hence $H \trianglelefteq G$.

($\Leftarrow$): Suppose that H is normal.

Let $h, h' \in H$ and let $a, b \in G$.

Then $H \trianglelefteq G \Rightarrow \exists h'' \in H$ s.t. $hb = bh''$

So $ahbh' = abh''h'$ which implies $ahbh' \in (ab)H$.

Then $(ahbh')H \cap (ab)H \neq \emptyset$.
So we must have $(ahbh')H = (ab)H$.
Since $a, b \in G$ and $h, h' \in H$ are arbitrary,
this shows that $(*)$ is well-defined. #

Cor Let $H \triangleleft G$. Then the cosets of H form a group G/H, called the quotient group of G by H, under the binary operation $(*)$.

Pf: Associativity follows from that in G. $H = eH$ is the identity.
The inverse of aH is given by $a^{-1}H$. #

Examples: • If G is an abelian group, then any subgroup is normal.

e.g. $n\mathbb{Z} \lhd \mathbb{Z}$, $\mathbb{Z} \lhd \mathbb{R}$ and $W \lhd V$.

$\mathbb{Z} \lhd \mathbb{Q} \lhd \mathbb{R} \lhd \mathbb{C}$, and

$W < V$ for any vector subspace $W \subset V$

Rmk V/W can be equipped with the structure of a vector space, called the quotient space of V by W.

• $\{e\} \lhd G$, $G \lhd G$. $G/\{e\} \cong G$, $G/G \cong \{e\}$.

• (i) $A_n \lhd S_n$ for any positive integer n.

(ii) Let $a \in D_n$ (the n-th dihedral group) be the rotation by $\frac{2\pi}{n}$. Then the cyclic subgroup $\langle a \rangle$ in D_n is normal.

(iii) $SO_n(\mathbb{R}) \triangleleft O_n(\mathbb{R})$

More generally, if $H < G$ has index $[G:H] = 2$, then $H \triangleleft G$
(reason: $\forall a \in G \setminus H$, we have $H \sqcup aH = G = H \sqcup Ha \Rightarrow aH = Ha$.)
In this case, $G/H \cong \mathbb{Z}_2$.

- $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$

- (Non-example)
For $S_3 = \{id, \rho, \rho^2, \mu, \rho\mu, \rho^2\mu\}$ where $\rho = (1\ 2\ 3)$, $\mu = (1\ 2)$, and $H = \langle \mu \rangle < S_3$. We've seen that $\rho H \neq H\rho$ and $\rho^2 H \neq H\rho^2$.
So H is not normal in S_3.

Kernel of a homomorphism

Prop Let $\phi: G \to G'$ be a homomorphism.

(1) $\phi(e_G) = e_{G'}$

(2) $\forall a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.

(3) $N \triangleleft G \Rightarrow \phi(N) \triangleleft \phi(G)$.

(4) $M \triangleleft G' \Rightarrow \phi^{-1}(M) \triangleleft G$.

Pf: (3) Suppose $N \triangleleft G$. We already know that $\phi(N) < G'$.

Now $\phi(g)\phi(N)\phi(g)^{-1} = \phi(gNg^{-1}) = \phi(N)$ $\forall g \in G$. So $\phi(N) \triangleleft \phi(G)$.

(4) Suppose $M \triangleleft G'$. Again we know that $\phi^{-1}(M) < G$.

Now $\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g)^{-1} \in M$ $\forall g \in G$ and $a \in \phi^{-1}(M)$.

So $g\phi^{-1}(M)g^{-1} \subset \phi^{-1}(M)$ $\forall g \in G$. Hence $\phi^{-1}(M) \triangleleft G$. #

Cor For any homomorphism $\phi: G \to G'$, $\ker(\phi) \triangleleft G$.

Examples • Recall that $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_n$

$k \longmapsto r :=$ remainder of k when it's divided by n

is a surjective homomorphism. $\ker(\phi) = n\mathbb{Z}$.

• $\phi : \mathbb{R} \longrightarrow U(1)$

$t \longmapsto e^{2\pi i t}$

is a surjective homomorphism. $\ker(\phi) = \mathbb{Z}$.

• $\phi : \mathbb{C}^\times \longrightarrow \mathbb{R}_{>0}$ (use multiplication for both $\mathbb{C}^\times$ & $\mathbb{R}_{>0}$)

$z \longmapsto |z|$

is a surjective homomorphism.

$\ker(\phi) = \{ z \in \mathbb{C}^\times \mid |z| = 1 \} = U(1) < \mathbb{C}^\times$.

• $\det : GL_n(F) \longrightarrow F^\times$ is a surjective homomorphism. $\ker = SL_n(F)$.

Prop Let $H \triangleleft G$. Then the map $\pi : G \to G/H$ defined by

$$\pi(a) := aH$$

is a homomorphism with $\ker(\pi) = H$.

Pf: $\forall a, b \in G$, $\pi(ab) = (ab)H = (aH)(bH) = \pi(a)\pi(b)$.

So π is a homomorphism.

$\ker(\pi) = \{a \in G \mid aH = H\} = H$. #

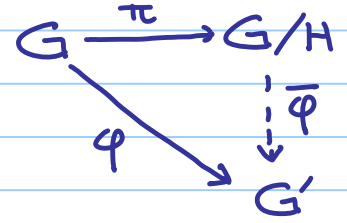
Hence any normal subgroup is the kernel of a homomorphism.

$\pi : G \to G/H$ is called the projection map or canonical map.

Thm Let $\varphi: G \to G'$ be a homomorphism. Let $H = \ker(\varphi)$. Then the map $\overline{\varphi}: G/H \to \varphi(G)$ defined by

$$\overline{\varphi}(aH) = \varphi(a)$$

is an isomorphism such that $\varphi = \overline{\varphi} \circ \pi$.



Pf: First of all, we need to show that $\overline{\varphi}$ is well-defined.

If $aH = a'H$, then $a = a'h$ for some $h \in H$.

So $\overline{\varphi}(aH) = \varphi(a) = \varphi(a'h) = \varphi(a')\varphi(h) = \varphi(a') = \overline{\varphi}(a'H)$. ($\varphi(h) = e'$)

Hence, $\overline{\varphi}$ is well-defined.

Since φ is a homomorphism, we have

$$\overline{\varphi}((aH)(bH)) = \overline{\varphi}((ab)H) = \varphi(ab) = \varphi(a)\varphi(b) = \overline{\varphi}(aH)\,\overline{\varphi}(bH)$$

Thus $\bar{\varphi}$ is a homomorphism.

$$\mathrm{Ker}(\bar{\varphi}) = \{aH \in G/H \mid \varphi(a) = e'\} = \{aH \in G/H \mid a \in H\} = H.$$

So $\bar{\varphi}$ is 1-1. It is clearly onto since $\forall \varphi(a) \in \varphi(G)$, $\bar{\varphi}(aH) = \varphi(a)$.
Hence $\bar{\varphi}$ is an isomorphism.

Finally, $\forall\, a \in G$, we have $(\bar{\varphi} \circ \pi)(a) = \bar{\varphi}(\pi(a)) = \bar{\varphi}(aH) = \varphi(a)$. #

Thus, any group homomorphism $\varphi : G \to G'$ can be decomposed as

$$G \twoheadrightarrow G/\mathrm{Ker}\,\varphi \xrightarrow[\bar{\varphi}]{\sim} \mathrm{Im}\,\varphi \hookrightarrow G' \qquad (\varphi : G \to G')$$

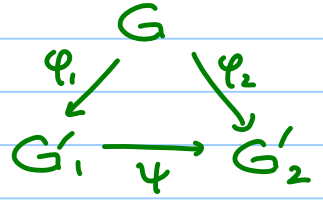
(Actually, any set-theoretic map $\varphi : A \to B$ can be decomposed as

$$A \twoheadrightarrow \mathrm{Im}\,\varphi \hookrightarrow B. \qquad (\varphi : A \to B)$$
)

Rmk: This is usually called the First Isomorphism Theorem, and is a useful tool in establishing isomorphisms.

Rmk: Given a group G and the equivalence relation $\sim_L$ defined by $N \trianglelefteq G$, we can consider the category with

- objects = pairs (G', φ), where H is a group and $\varphi : G \to G'$ is a homomorphism s.t. $\varphi(a) = \varphi(b)$ whenever $a \sim_L b$.
- morphisms are comm. diagrams



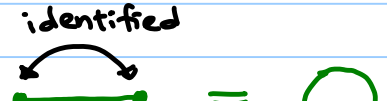
where ψ is a homomorphism

Then the pair $(G/N, \pi)$ is an initial object in this category.

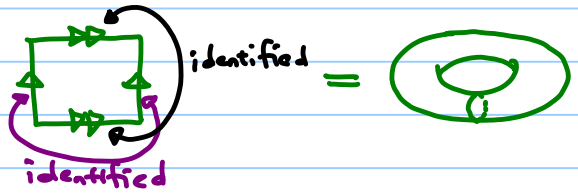
e.g.
- $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

  (The isom. is given by mapping $k+n\mathbb{Z}$ to k for $k = 0, 1, \ldots, n-1$.)
- $\mathbb{R}/\mathbb{Z} \cong U(1)$

  (The isom. is $\mathbb{R}/\mathbb{Z} \longrightarrow U(1)$, $t+\mathbb{Z} \longmapsto e^{2\pi i t}$)



  For $\mathbb{Z}^2 \lhd \mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z}^2$ is a torus



  (and similarly for higher dimensions)
- $\mathbb{C}^\times/U(1) \cong \mathbb{R}_{>0}$
- $GL_n(F)/SL_n(F) \cong F^\times$

Example (Converse of Theorem of Lagrange is false)

We claim that A_4 has no subgroup of order 6.

Pf: Suppose that $\exists\, H < A_4$ with $|H| = 6$.

Then $H \triangleleft A_4$ and $A_4/H \cong \mathbb{Z}_2$.

This implies that $\forall \sigma \in A_4$, $(\sigma H)(\sigma H) = H$.

In other words, $\sigma^2 \in H \quad \forall \sigma \in A_4$.

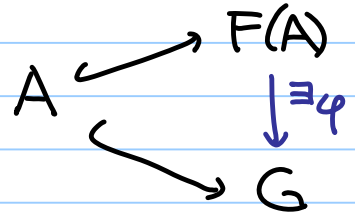
Now $(p,q,r) = (p,r,q)^2$ $\forall$ distinct $p,q,r \in \{1,2,3,4\}$.

There are 8 such elements and they are all inside H,

contradicting $|H| = 6$.

Presentations

Let G be a group. Then we can always find a subset $A \subset G$ s.t. $G = \langle A \rangle$ (at least we can take $A = G$). By the universal property of free groups, $\exists!$ group homomorphism $\varphi : F(A) \longrightarrow G$, where $F(A)$ is the free group on A, s.t. the following diagram commutes:



So every group (resp. abelian group) is a quotient of a free group (resp. free abelian group).

Def A presentation of a group G is an (explicit) isomorphism

$$G \cong F(A)/R$$

where $F(A)$ is the free group on a set A.
Elements of A are called generators and elements of R are called relations.

If we can find a finite set A and a finite set $R \subset F(A)$ of words s.t. $R = \langle R \rangle$, then we say that $G \cong F(A)/R$ is finitely presented. In this case, we write $G = \langle A | R \rangle$.

Rmk A group can have many different presentations.

e.g.
- $F(A) = \langle A \mid \emptyset \rangle$
- $\mathbb{Z}_n = \langle 1 \mid n \rangle$
- $D_n = \langle a, b \mid a^n, b^2, abab \rangle$
- $S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, \sigma\tau\sigma\tau \rangle$