THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 5
Due Date: 17th October 2024

Compulsory Part

1. Let K and L be normal subgroups of G with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

2. Suppose
$$\{e\} \xrightarrow{\iota} N \longrightarrow G \xrightarrow{\varphi} K \to \{e\}$$
is an exact sequence of groups. Suppose also that there is a group homomorphism $\tau : G \to N$ such that $\tau \circ \iota = \mathrm{id}_N$. Prove that $G \simeq N \times K$.

3. Show that if
$$H_0 = \{e\} \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$$
is a subnormal (normal) series for a group G, and if H_{i+1}/H_i is of finite order s_{i+1}, then G is of finite order $s_1 s_2 \cdots s_n$.

4. Show that an infinite abelian group can have no composition series.

   [*Hint:* Use the preceding exercise, together with the fact that an infinite abelian group always has a proper normal subgroup.]

5. Show that a finite direct product of solvable groups is solvable.

Optional Part

1. Suppose N is a normal subgroup of a group G of prime index p. Show that, for any subgroup $H \leq G$, we either have
   - $H \leq N$, or
   - $G = HN$ and $[H : H \cap N] = p$.
2. Suppose N is a normal subgroup of a group G such that $N \cap [G, G] = \{e\}$. Show that $N \leq Z(G)$.
3. Let $H_0 = \{e\} \leq H_1 \leq \cdots \leq H_n = G$ be a composition series for a group G. Let N be a normal subgroup of G, and suppose that N is a simple group. Show that the distinct groups among $H_0, H_i N$ for $i = 0, \cdots, n$ also form a composition series for G.

   [***Hint:*** Note that $H_i N$ is a group. Show that $H_{i-1}N$ is normal in $H_i N$. Then we have
   $$(H_i N)/(H_{i-1}N) \simeq H_i/[H_i \cap (H_{i-1}N)],$$
   and the latter group is isomorphic to
   $$[H_i/H_{i-1}]/[(H_i \cap (H_{i-1}N))/H_{i-1}].$$
   But H_i/H_{i-1} is simple.]
4. If H is a maximal proper subgroup of a finite solvable group G, prove that $[G : H]$ is a prime power.