THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 4
Due Date: 10th October 2024

Compulsory Part

1. Show that the center of a direct product is the direct product of the centers, i.e.

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

   Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.

2. Show that if G is nonabelian, then the quotient group $G/Z(G)$ is not cyclic.

   [*Hint:* Show the equivalent contrapositive, namely, that if $G/Z(G)$ is cyclic then G is abelian (and hence $Z(G) = G$).]

3. Using the preceding question, show that a nonabelian group G of order pq where p and q are primes has a trivial center.

4. Let N be a normal subgroup of G and let H be any subgroup of G. Let $HN = \{hn \mid h \in H, n \in N\}$. Show that HN is a subgroup of G, and is the smallest subgroup containing both N and H.

5. Show directly from the definition of a normal subgroup that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H.

6. Let H, K, and L be normal subgroups of G with $H \leq K \leq L$. Let $A = G/H$, $B = K/H$, and $C = L/H$.

   (a) Show that B and C are normal subgroups of A, and $B \leq C$.

   (b) To what quotient group of G is $(A/B)/(C/B)$ isomorphic?

Optional Part

1. Let F be a field, and $n \in \mathbb{Z}_{>0}$.
   (a) Show that $SL_n(F)$ is a normal subgroup in $GL_n(F)$.
   (b) When F is a finite field, show that $[GL_n(F) : SL_n(F)] = |F| - 1$.
2. Let $F = F^A$ be the free group on two generators $A = \{a, b\}$. Show that the normal subgroup generated by the single commutator $aba^{-1}b^{-1}$ is the commutator of F.
3. Show that the converse to the Theorem of Lagrange holds for an abelian group, namely, if G is a finite abelian group and $d \mid |G|$, then there exists a subgroup of G of order d.
4. Prove that A_n is simple for $n \geq 5$, following the steps and hints given.
   (a) Show that A_n contains every 3-cycle if $n \geq 3$.
   (b) Show that A_n is generated by the 3-cycles for $n \geq 3$ [*Hint:* Note that $(a, b)(c, d) = (a, c, b)(a, c, d)$ and $(a, c)(a, b) = (a, b, c)$.]
   (c) Let r and s be fixed elements of $\{1, 2, \cdots, n\}$ for $n \geq 3$. Show that A_n is generated by the n "special" 3-cycles of the form (r, s, i) for $1 \leq i \leq n$. [*Hint:* Show every 3-cycle is the product of "special" 3-cycles by computing

$$(r, s, i)^2, \ (r, s, j)(r, s, i)^2, \ (r, s, j)^2(r, s, i),$$

   and

$$(r, s, i)^2(r, s, k)(r, s, j)^2(r, s, i).$$

   Observe that these products give all possible types of 3-cycles.]
   (d) Let N be a normal subgroup of A_n for $n \geq 3$. Show that if N contains a 3-cycle, then $N = A_n$. [*Hint:* Show that $(r, s, i) \in N$ implies that $(r, s, j) \in N$ for $j = 1, 2, \cdots, n$ by computing

$$((r, s)(i, j))(r, s, i)^2((r, s)(i, j))^{-1}.]$$

   (e) Let N be a nontrivial normal subgroup of A_n for $n \geq 5$. Show that one of the following cases must hold, and conclude in each case that $N = A_n$.

   Case I N contains a 3-cycle.

   Case II N contains a product of disjoint cycles, at least one of which has length greater than 3. [*Hint:* Suppose N contains the disjoint product $\sigma = \mu(a_1, a_2, \cdots, a_r)$. Show $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it.]

   Case III N contains a disjoint product of the form $\sigma = \mu(a_4, a_5, a_6)(a_1, a_2, a_3)$. [*Hint:* Show $\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1}$ is in N, and compute it.]

   Case IV N contains a disjoint product of the form $\sigma = \mu(a_1, a_2, a_3)$ where μ is a product of disjoint 2-cycles. [*Hint:* Show $\sigma^2 \in N$ and compute it.]

   Case V N contains a disjoint product σ of the form $\sigma = \mu(a_3, a_4)(a_1, a_2)$, where μ is a product of an even number of disjoint 2-cycles.
   [*Hint:* Show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it to deduce that $\alpha = (a_2, a_4)(a_1, a_3)$ is in N. Using $n \geq 5$ for the first time, find $i \neq a_1, a_2, a_3, a_4$ in $\{1, 2, \cdots, n\}$. Let $\beta = (a_1, a_3, i)$. Show that $\beta^{-1}\alpha\beta\alpha \in N$, and compute it.]