MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 7

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-11-01 (Friday) 23:59****.*

1. Let T be a linear operator on a finite-dimensional vector space V.
   (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
   (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

2. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-cyclic subspace of V generated by a nonzero vector v. Let $k = \dim(W)$. Prove the following statements.
   (a) $\{x, T(x), T^2(x), ..., T^{k-1}(x)\}$ is a basis for W.
   (b) If $a_0x + a_1T(x) + a_2T^2(x) + ... + a_{k-1}T^{k-1}(x) + T^k(x) = 0$, then $f_{T_W}(t) = (-1)^k(a_0 + a_1t + a_2t^2 + ... + a_{k-1}t^{k-1} + t^k)$.

3. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself (i.e. there exists $x \in V$ such that V is the T-cyclic subspace generated by x). Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t)$.

4. Let V be an inner product space over F. Prove the following statements.
   (a) If x, y are orthogonal, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
   (b) *Parallelogram law*: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.
   (c) Let $v_1, v_2, ..., v_k$ be an orthogonal set in V, and let $a_1, a_2, ..., a_k \in F$, then

$$\|\sum_{i=1}^{k} a_i v_i\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

5. Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

Exercises

The following are extra recommended exercises not included in the homework.

1. Let T be a linear operator on a vector space V, and let $W_1, W_2, ..., W_k$ be T-invariant subspaces of V.

   (a) Prove that $\bigcap_{i=1}^{k} W_i$ is T-invariant.

   (b) Prove that $W_1 + W_2 + \cdots W_k$ is T-invariant.

   (c) Suppose V is finite-dimensional and $V = \bigoplus_{i=1}^{k} W_i$. Prove that T is diagonalizable if and only if T_{W_i} is diagonalizable for all i.

2. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $T = cI_V$ for some $c \in F$.

3. Let $\mathcal{C}$ be a collection of diagonalizable linear operators on a finite-dimensional vector space V. Prove that there is an ordered basis β such that $[T]_\beta$ is a diagonal matrix for all $T \in \mathcal{C}$ if and only if the operators of $\mathcal{C}$ commute under composition.

4. Use the Cayley-Hamilton theorem to prove its corollary for matrices.

5. Let A be an $n \times n$ matrix. Prove that $\dim(\text{span}(\{I_n, A, A^2, ...\})) \leq n$.

6. In $\mathbb{C}^2$, show that $\langle x, y\rangle = xA^*y$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

   Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

7. Suppose $\langle \cdot, \cdot\rangle_1$ and $\langle \cdot, \cdot\rangle_2$ are two inner products in a vector space V. Prove that $\langle \cdot, \cdot\rangle = \langle \cdot, \cdot\rangle_1 + \langle \cdot, \cdot\rangle_2$ is another inner product on V.

8. Let V be a vector space over F, where $F = \mathbb{R}$ or $\mathbb{C}$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot\rangle$. If $T : V \to W$ in linear, prove that $\langle x, y\rangle' = \langle T(x), T(y)\rangle$ defines an inner product on V if and only if T is one-to-one.