Lecture 9:

Recall: Invertibility and Isomorphism

Let V and W be vector spaces and let $T: V \to W$ be linear.

- T is invertible : $\exists\, T^{-1}: W \to V$ such that:

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \circ T^{-1} = I_W$$

($\therefore$ T is bijective)

- If T is invertible, T^{-1} is linear.

Pf: Let $\vec{w}_1, \vec{w}_2 \in W$ and $c \in F$.

$\because$ T is invertible $\therefore$ $\exists\, \vec{v}_1, \vec{v}_2$ such that $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$.

Then: $T^{-1}(c\vec{w}_1 + \vec{w}_2) = T^{-1}(cT(\vec{v}_1) + T(\vec{v}_2)) = T^{-1}(T(c\vec{v}_1 + \vec{v}_2))$ ($\because$ T is linear)

$$= c_1 \vec{v}_1 + \vec{v}_2$$

$$= c_1 T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2)$$

$\therefore$ T^{-1} is linear.

- Suppose $T: V \to W$ is invertible. Then:

$$\dim(V) < +\infty \text{ iff } \dim(W) < +\infty$$

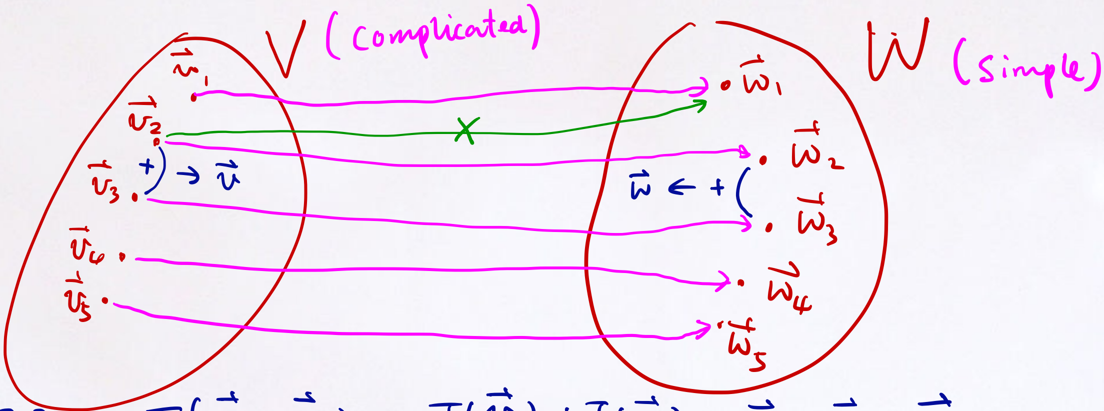
And in this case, $\dim(V) = \dim(W)$

- Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Then: T is invertible iff $[T]_\beta^\gamma$ is invertible.

Also, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$

Definition: Let V and W be two vector spaces.
We say V is isomorphic to W if $\exists$ an invertible linear transformation $T: V \to W$.
In this case, T is called an isomorphism from V onto W.



$$\vec{v}_2+\vec{v}_3 = \vec{v}$$

$$T(\vec{v}) = \vec{w} \; ??$$

$$T(\vec{v}_2+\vec{v}_3) = T(\vec{v}_2)+T(\vec{v}_3) = \vec{w}_2+\vec{w}_3 = \vec{w}$$

Thm: Let V and W be finite-dimensional vector spaces.

Then: V is isomorphic to W iff $\dim(V) = \dim(W)$.

Proof: $(\Rightarrow)$ This direction follows from previous Lemma.

$(\Leftarrow)$: Suppose $\dim(V) = \dim(W) \stackrel{\text{def}}{=} n$ and let

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be basis for V;

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ be basis for W.

Then $\exists$ linear $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$

for $i = 1, 2, \dots, n$.

By construction, T is onto and $\dim(V) = \dim(W)$.

So, T is one-to-one. $\therefore$ T is invertible.

<u>Corollary</u>: Let V be a vector space over F.

Then: V is isomorphic to F^n iff $\dim(V) = n$

Space of linear transformation

Prop: Let V and W be vector spaces over F.

Then: the set $\mathcal{L}(V, W)$ of all linear transformations from V to W is a vector space over F under the following operations: for linear $T, U : V \to W$,

we define: $(T+U) : V \to W$ by $(T+U)(\underset{V}{\vec{x}}) = T(\vec{x}) + U(\vec{x})$

and for any $a \in F$, we define $aT : V \to W$ by

$$(aT)(\underset{V}{\vec{x}}) = a\,T(\vec{x})$$

Thm: Let V and W be finite-dimensional vector spaces over F. with dimension n and m respectively. Let β and γ be the ordered bases for V and W respectively.

Then: the map $\Phi : \mathcal{L}(V,W) \longrightarrow M_{m \times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Cor: $\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W) = nm$.

Proof: Φ is linear : $\Phi(T+U) = [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

$= \Phi(T) + \Phi(U)$

$\Phi(aT) = [aT]_\beta^\gamma = a[T]_\beta^\gamma$

$= a\Phi(T)$.

Φ is bijective :

For any $A = (A_{ij}) \in M_{m\times n}(F)$, ~~want to show that~~

$\exists!\, T : V \to W$ such that $\cancel{\Phi(T) = [T]_\beta^\gamma} = A$.

$$T(\vec{v}_j) = \sum_{i=1}^{m} A_{ij}\vec{w}_i \quad \text{for } j = 1, 2, \ldots, m$$

$$\begin{pmatrix} | \\ [T(\vec{v}_j)]_\gamma \\ | \end{pmatrix}$$

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$

$\Downarrow$

$\therefore$ For any $A \in M_{m\times n}(F)$, $\exists!\, T : V \to W$ such that $\Phi(T) = A$. (Onto)

(1-1)

$\therefore$ Φ is bijective.

Def Let β be the ordered basis for an n-dimensional vector space V over F. The map $\phi_\beta : V \to F^n$, $\vec{x} \longmapsto [x]_\beta$ is called standard representation of V with respect to β.

Prop: ϕ_β is an isomorphism.

Given vector spaces V and W of dimension n and m, with ordered bases β and γ respectively. Then, for any $T: V \to W$ (linear), we have:

$$
\begin{tikzcd}
\vec{v} \in V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \ni T(\vec{v}) := \vec{w} \arrow[d, "\phi_\gamma"] \\
[\vec{v}]_\beta \in F^n \arrow[r, "L_A"] & F^m \quad [\vec{w}]_\gamma = [T(\vec{v})]_\gamma
\end{tikzcd}
$$

$$\Rightarrow \quad \phi_\gamma \circ T(\vec{v}) = L_A \circ \phi_\beta(\vec{v})$$

where $A = [T]_\beta^\gamma$

$$\Leftrightarrow \quad [T(\vec{v})]_\gamma = [T]_\beta^\gamma [\vec{v}]_\beta$$

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$.

$$V_{\beta'} \xrightarrow{I_V} V_{\beta}$$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}$

Proof: (a) Since I_V is invertible, Q is invertible.

b) Let $\vec{v} \in V$. Then: $[\vec{v}]_\beta = [I_V(\vec{v})]_\beta = \underbrace{[I_V]_{\beta'}^{\beta}}_{= Q}[\vec{v}]_{\beta'}$

Def: The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the change of coordinate matrix from β' to β.

<u>Remark</u>: To compute $Q = [I_V]_{\beta'}^{\beta}$,

if $\beta = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ and $\beta' = \{\vec{x}_1{}', \vec{x}_2{}', \ldots, \vec{x}_n{}'\}$,

then:

$$Q = \begin{pmatrix} | & & \\ [I_V(\vec{x}_1{}')]_\beta & - & - \,- \\ | & & \end{pmatrix}$$

$$= \begin{pmatrix} | & & | & \\ [\vec{x}_1{}']_\beta & \cdots & [\vec{x}_j{}']_\beta & \cdots \\ | & & | & \end{pmatrix}$$

Proposition: Let T be a linear operator on finite-dim V. Let β and β' be ordered bases of V. Suppose $Q = [I_V]_{\beta'}^{\beta}$.

Then:
$$[T]_{\beta'} = Q^{-1}[T]_{\beta}\,Q$$

$$\overset{\beta}{V} \xrightarrow{T} \overset{\beta}{V} \rightsquigarrow [T]_{\beta}$$

$$\overset{\beta'}{V} \xrightarrow{T} \overset{\beta'}{V} \rightsquigarrow [T]_{\beta'}$$

Proof:
$$\begin{aligned} Q[T]_{\beta'} &= [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [I_V \circ T]_{\beta'}^{\beta} \\ &= [T \circ I_V]_{\beta'}^{\beta} \\ &= [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} \\ &= [T]_{\beta}\,Q \end{aligned}$$

$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V}$$

Remark: A linear $T: V \to V$ is called linear operator.

Corollary: Let $A \in M_{n\times n}(F)$ and let $\gamma = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ be an ordered basis for F^n.

Then: $[L_A]_\gamma = Q^{-1} A Q$, $Q = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$

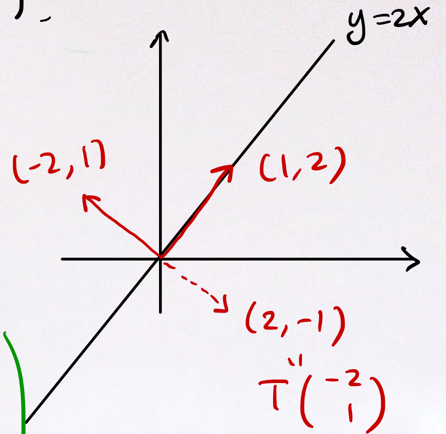
$$\iff [L_A]_\gamma = Q^{-1} [L_A]_\beta Q$$

standard ordered basis.

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y=2x$.

Want to compute $[T]_\beta$, where $\beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$.

Consider $\beta' = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix} \right\}$ for $\mathbb{R}^2$



- $$[T]_{\beta'} = \left(\left[T\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} \;\; \left[T\begin{pmatrix}-2\\1\end{pmatrix}\right]_{\beta'} \right) = \left(\left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} \;\; \left[\begin{pmatrix}2\\-1\end{pmatrix}\right]_{\beta'} \right) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- $$Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix} \Rightarrow Q^{-1} = \frac{1}{5}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$$

$\therefore \quad [T]_{\beta'} = Q^{-1} [T]_{\beta} Q$

$\Longleftrightarrow \quad [T]_{\beta} = Q \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{[T]_{\beta'}}} Q^{-1} = \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$

<u>Def</u>: Given two matrices $A, B \in M_{n\times n}(F)$.
We say B is <u>similar</u> to A if $\exists\, Q \in M_{n\times n}$ s.t.

$$B = Q^{-1} A Q.$$