Lecture 8

Recall:

Matrix representation

Notation: An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

(e.g. $\mathbb{R}^2$ $\quad \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{\beta_1} \neq \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}_{\beta_2}$ as ordered basis)

Definition: Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists !\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector $[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$

(F^n)

Recall:

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for V

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ for W

$$[T]_\beta^\gamma = \begin{array}{c} m \left[\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix} \right. \\ \underbrace{\hphantom{[T(\vec{v}_1)]_\gamma \quad [T(\vec{v}_2)]_\gamma \quad \cdots \quad [T(\vec{v}_n)]_\gamma}}_{n} \end{array}$$

$\in$

$M_{m \times n}$

Recall:

Composition of linear transformations and matrix multiplication

Thm: Let V and W be two vector spaces over the same field F.

And let $T: V \to W$ and $U: W \to Z$ be linear.

$$\begin{tikzcd} V_{\substack{\alpha \\ n}} \arrow[r, "T"] \arrow[rr, bend left, "UT"] & W_{\substack{\beta \\ m}} \arrow[r, "U"] & Z_{\substack{\gamma \\ p}} \end{tikzcd}$$

(i) Then the composition $UT: V \to Z$ is linear.

(ii) If V, W, Z have ordered bases α, β, γ respectively,

then:
$$\underbrace{[UT]_\alpha^\gamma}_{M_{p\times n}} = \underbrace{\underbrace{[U]_\beta^\gamma}_{M_{p\times m}} [T]_\alpha^\beta}_{\text{matrix multiplication.}} \in M_{m\times n}$$

Recall:

Corollary: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Let $T: V \to W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$\underbrace{[\underset{W}{T(\vec{u})}]_\gamma}_{\text{Lin. Transf.}} = \underbrace{[T]_\beta^\gamma \, [\vec{u}]_\beta}_{\text{Matrix multiplication}}$$

Example: $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(A) \overset{def}{=} A^T + 2A.$$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$.

$$[T(B)]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} [B]_\beta = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 4 \\ 0 \end{pmatrix}$$

$$T(B) = \begin{pmatrix} 9 & 5 \\ 4 & 0 \end{pmatrix}$$

Invertibility and isomorphism

Lemma: Suppose $T: V \to W$ is invertible.

(T is invertible means there exists a linear transformation $T^{-1}: W \to V$ such that $T \circ T^{-1} = I_W$ and $T^{-1} \circ T = I_V$)

Then: $\dim(V) < +\infty$ iff $\dim(W) < +\infty$

And in this case, $\dim(V) = \dim(W)$

Remark: If T is linear and invertible, T^{-1} is also linear.

Pf: Let $\vec{w}_1, \vec{w}_2 \in W$ and $c \in F$.

$\because$ T is invertible $\therefore$ $\exists \vec{v}_1, \vec{v}_2$ such that $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$.

Then: $T^{-1}(c\vec{w}_1 + \vec{w}_2) = T^{-1}(cT(\vec{v}_1) + T(\vec{v}_2)) = T^{-1}(T(c\vec{v}_1 + \vec{v}_2))$ ($\because$ T is linear)

$$= c_1 \vec{v}_1 + \vec{v}_2$$

$$= c_1 T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2)$$

$\therefore$ T^{-1} is linear.

<u>Proof:</u> Suppose $\dim(V) = n < +\infty$ and $\beta = \{\vec{x}_1, \ldots, \vec{x}_n\}$ is a basis for V. Then:

$$W = R(T) = \operatorname{span}\{T(\beta)\} = \operatorname{span}\underbrace{\{T(\vec{x}_1), \ldots, T(\vec{x}_n)\}}_{n \text{ elements}}$$

$$\therefore \quad \dim(W) \leq n = \dim(V) < +\infty$$

Apply the same argument to T^{-1} to show that

$$\dim(V) \leq \dim(W)$$

In particular, if $\dim(V) < +\infty$ and $\dim(W) < +\infty$ then: $\dim(V) \leq \dim(W)$ and $\dim(W) \leq \dim(V) \iff \dim(W) = \dim(V)$

Proposition: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Let $T: V \to W$ be linear transformation.

Then. T is invertible iff $[T]_\beta^\gamma$ is invertible.

Furthermore, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$

<u>Proof:</u> Suppose T is invertible. Then: $\dim(V) = \dim(W) = n$

Since $T \circ T^{-1} = I_W$, $I_n = [I_W]_\gamma = [T \circ T^{-1}]_\gamma$

$$\underset{\gamma}{W} \xrightarrow{T^{-1}} \underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W}$$

$$I_n = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$$

Similarly, $T^{-1} \circ T = I_V$. $I_n = [I_V]_\beta = [T^{-1} \circ T]_\beta$

$$I_n = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

$\therefore$ $[T]_\beta^\gamma$ is invertible and $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$.

Conversely, suppose $A := [T]_\beta^\gamma$ is invertible. $(\Rightarrow \dim(V) = \dim(W))$

$\because \dim(V) = \dim(W)$

$\therefore$ We only need to show T is one-to-one.

So, suppose $T(\vec{x}_1) = T(\vec{x}_2)$

$$\Leftrightarrow [T(\vec{x}_1)]_\gamma = [T(\vec{x}_2)]_\gamma$$

$$\Rightarrow \underbrace{[T]_\beta^\gamma}_{=A} [\vec{x}_1]_\beta = \underbrace{[T]_\beta^\gamma}_{=A} [\vec{x}_2]_\beta$$

$$\Rightarrow [\vec{x}_1]_\beta = [\vec{x}_2]_\beta \Rightarrow \vec{x}_1 = \vec{x}_2$$

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Corollary: Let V be a finite-dimensional vector space with ordered basis β. Let $T: V \to V$ be a linear transformation.

Then: T is invertible iff $[T]_\beta$ is invertible

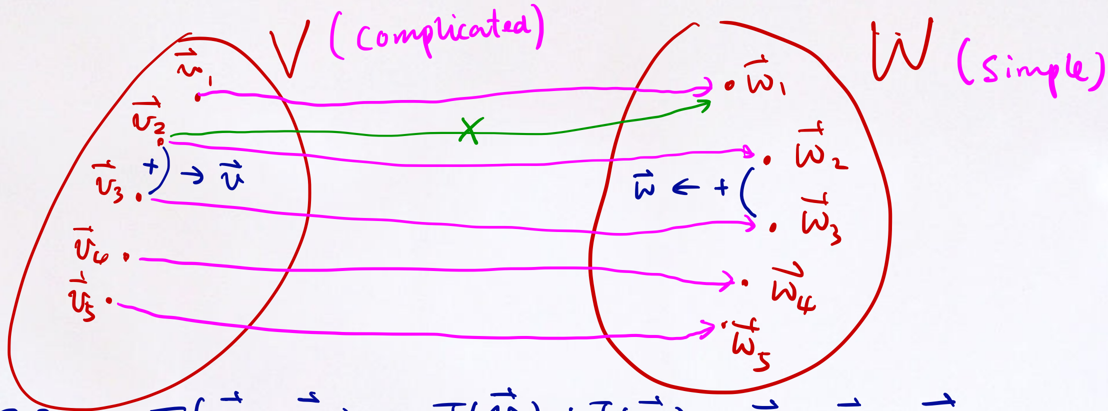
Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

$[L_A]_\beta$ ← standard ordered basis

Corollary: Let $A \in M_{n \times n}(F)$. Then: A is invertible iff L_A is invertible. $(L_A)^{-1} = L_{A^{-1}}$

$$\left(\begin{array}{l} [(L_A)^{-1}]_\beta = ([L_A]_\beta)^{-1} = A^{-1} = [L_{A^{-1}}]_\beta \\ \therefore (L_A)^{-1} = L_{A^{-1}} \end{array} \right)$$

Definition: Let V and W be two vector spaces.
We say V is isomorphic to W if $\exists$ an invertible linear transformation $T: V \to W$.
In this case, T is called an isomorphism from V onto W.



$$\vec{v}_2+\vec{v}_3$$
$$\|$$
$$T(\vec{v}) = \vec{w} \ ??$$

$$T(\vec{v}_2+\vec{v}_3) = T(\vec{v}_2)+T(\vec{v}_3) = \vec{w}_2+\vec{w}_3 = \vec{w}$$