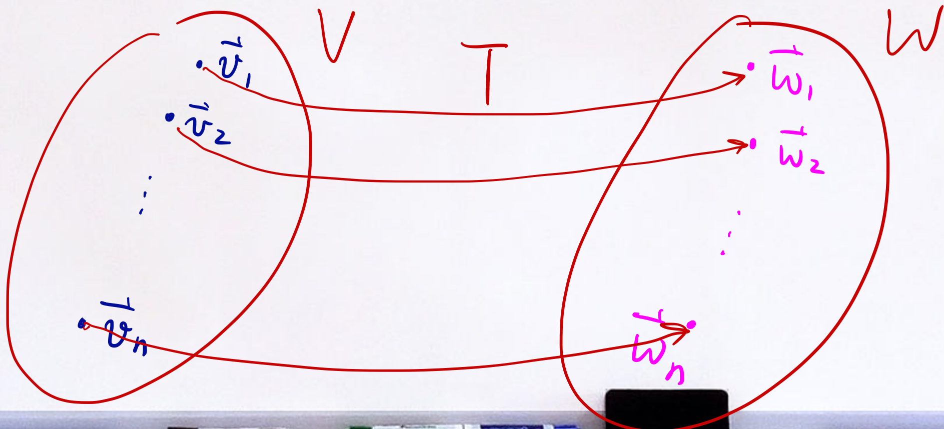
Lecture 7

Recall

Thm: Let V and W be vector spaces. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. Then, given any $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$. $\exists$ a unique linear transformation $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$



Corollary: Let V be a vector space with a finite basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

Then any linear transformation from V to another vector space W is completely determined by its values on β.

(That is, if U and T are linear transformations from V to W s.t. $U(\vec{v}_i) = T(\vec{v}_i)$, then $U = T$)

Matrix representation

Notation: An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

(e.g. $\mathbb{R}^2$ $\quad \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{\beta_1} \neq \underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}_{\beta_2}$ as ordered basis)

Definition: Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists !\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector (F^n)

$$[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

Remark: Define a map $V \to F^n$. This map is linear
$$\vec{x} \mapsto [\vec{x}]_\beta$$

(HW. $[\underbrace{a\vec{x}+\vec{y}}_{\in V}]_\beta = a[\vec{x}]_\beta + [\vec{y}]_\beta$)

Now, suppose V and W are finite-dimensional vector spaces with ordered bases $\underbrace{\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}}_{\text{(for } V)}$ and $\underbrace{\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}}_{\text{(for } W)}$ respectively.

Let $T: V \to W$ be a linear transformation.

Then for each $1 \le j \le n$, $\exists\, a_{ij} \in F$ $1 \le i \le m$ such that

$$\underbrace{T(\vec{v}_j)}_{\in W} = \sum_{i=1}^{m} a_{ij}\vec{w}_i \quad \text{for} \quad 1 \le j \le n.$$

Definition: With this notation as above, we call the matrix

$$A \overset{\text{def}}{:=} (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$$

the matrix representation of T in the ordered bases β and γ, and denoted it as $A = [T]_\beta^\gamma$.

$$\underset{\underset{W}{\cap}}{T(\vec{v}_j)} = \sum_{i=1}^{m} a_{ij}\,\vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}} \end{pmatrix}$$

$$\begin{matrix} \| & \| & & \| \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & & [T(\vec{v}_n)]_\gamma \end{matrix}$$

$$\underset{\underset{W}{\cap}}{T(\vec{v}_1)} = \sum_{i=1}^{m} a_{i1}\,\vec{w}_i$$

$$\Downarrow$$

$$[T(\vec{v}_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in F^n$$

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for V

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ for W

$$[T]_\beta^\gamma = m\left[\underbrace{\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix}}_{n}\right.$$

$M_{m \times n}$

Examples:

- Let $A \in M_{m\times n}(F)$. $L_A : F^n \to F^m$ defined by: $L_A(\vec{x}) \overset{def}{=} A\vec{x}$, $\vec{x} \in F^n$

Let β and γ be the standard bases for F^n and F^m resp.

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \dots \right\}$

$$[L_A]_\beta^\gamma = \begin{pmatrix} | & & | \\ [A\vec{e}_1]_\gamma & \cdots & [A\vec{e}_n]_\gamma \\ | & & | \end{pmatrix}$$

first col. of A

n^{th} col of A

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{first col of } A$$

$$= \begin{pmatrix} | & & | \\ 1^{st} \text{ col of } A & & n^{th} \text{ col of } A \\ | & & | \end{pmatrix} = A$$

• For $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined as $T(f(x)) = f'(x)$.

Let $\beta = \{1, x, x^2, \dots, x^n\}$ be an ordered basis for $P_n(\mathbb{R})$

Let $\gamma = \{1, x, x^2, \dots, x^{n-1}\}$ be an ordered basis for $P_{n-1}(\mathbb{R})$

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [T(1)]_\gamma & [T(x)]_\gamma & \cdots & [T(x^n)]_\gamma \\ | & | & & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

$T(1) = 0$, $T(x) = 1$, $T(x^n) = nx^{n-1}$

<u>Example:</u> $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(A) \overset{\text{def}}{:=} A^T + 2A$$

tranpose

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ — ordered basis

$$T(\beta) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}$$

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

<u>Example:</u> $T: P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by :

$$T(f) \overset{def}{=} \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

Consider ordered basis : $\beta = \{1, x, x^2\}$ for $P_2(\mathbb{R})$

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$T(\beta) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_{4\times3}$$

Composition of linear transformations and matrix multiplication

Thm: Let V and W be two vector spaces over the same field F. And let $T: V \to W$ and $U: W \to Z$ be linear.

$$\begin{tikzcd} \underset{\alpha,\ n}{V} \arrow[r, "T"] \arrow[rr, bend left, "UT"] & \underset{\beta,\ m}{W} \arrow[r, "U"] & \underset{\gamma,\ p}{Z} \end{tikzcd}$$

(i) Then the composition $UT: V \to Z$ is linear.

(ii) If V, W, Z have ordered bases α, β, γ respectively, then:

$$\underbrace{[UT]_\alpha^\gamma}_{M_{p\times n}} = \underbrace{\underbrace{[U]_\beta^\gamma}_{M_{p\times m}} [T]_\alpha^\beta}_{\text{matrix multiplication}} \in M_{m\times n}$$

(i) Let $\vec{x}, \vec{y} \in V$ and $a \in F$. Then:

$$UT(a\vec{x} + \vec{y}) = U(aT(\vec{x}) + T(\vec{y})) = aUT(x) + UT(\vec{y})$$

$\therefore$ UT is linear.

(ii) Suppose

$$\alpha = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$$
$$\beta = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$$
$$\gamma = \{\vec{z}_1, \vec{z}_2, \dots, \vec{z}_p\}$$

$$[UT]_\alpha^\gamma = (C_{ij})_{\substack{1 \le i \le p \\ 1 \le j \le n}}$$

$$\underset{\substack{\uparrow \\ M_{p\times m}(F)}}{[U]_\beta^\gamma = A} \overset{\text{def}}{=} (a_{ik})_{\substack{1 \le i \le p \\ 1 \le k \le m}}$$ means: $$U(\vec{w}_k) = \sum_{i=1}^{p} a_{ik} \vec{z}_i$$

$1 \le k \le m$

$$= \begin{array}{c} \begin{pmatrix} & a_{1k} & \\ & a_{2k} & \\ & \vdots & \\ & a_{pk} & \end{pmatrix} \\ \uparrow \\ k\text{th} \end{array}$$

$$[T]_\alpha^\beta = \underset{\underset{M_{m\times n}(F)}{\cap}}{B} \overset{def}{=} (b_{kj})_{\substack{1\le k\le m\\ 1\le j\le n}} \quad \text{means} \quad T(\vec{v}_j) = \sum_{k=1}^{m} b_{kj}\, \vec{w}_k$$

for $1\le j\le n$

Then:

$$\begin{aligned} UT(\vec{v}_j) &= U\Big(\sum_{k=1}^{m} b_{kj}\, \vec{w}_k\Big) \\ &= \sum_{k=1}^{m} b_{kj}\, U(\vec{w}_k) \\ &= \sum_{k=1}^{m} b_{kj}\Big(\sum_{i=1}^{p} a_{ik}\, \vec{z}_i\Big) = \sum_{i=1}^{p} \boxed{\Big(\sum_{k=1}^{m} a_{ik} b_{kj}\Big)}\, \vec{z}_i \end{aligned}$$

(i, j)-entry of AB

So, $[UT]_\alpha^\gamma = AB = [U]_\beta^\gamma [T]_\alpha^\beta$

Corollary: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \to W$ be linear. Then: for any $\vec{u} \in V$, we have:

$$[\underbrace{T(\vec{u})}_{\substack{W \\ \text{Lin. Transf.}}}]_\gamma = \underbrace{[T]_\beta^\gamma [\vec{u}]_\beta}_{\text{Matrix multiplication}}$$

Proof: Fix $\vec{u} \in V$ and consider two linear transformations:

$$f: \overset{\alpha}{F} \to \overset{\beta}{V} \qquad\qquad g: F \to \overset{\gamma}{W}$$

defined by $\qquad\qquad$ defined by

$$f(\underset{F}{a}) = a\vec{u} \in V \qquad\qquad g(a) = a\,T(\vec{u}) \in W$$

f and g are linear transformations. Also, $g = T \circ f$.

Let $\alpha = \{1\}$ be the standard ordered basis for F.

$$[T(\vec{u})]_\gamma = [g(1)]_\gamma = \underset{T\circ f}{\underset{\shortparallel}{[g]}}{}_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma \begin{bmatrix} | \\ f(1) \end{bmatrix}_\beta$$

$$= [T]_\beta^\gamma \begin{bmatrix} | \\ \vec{u} \end{bmatrix}_\beta$$

$$\begin{tikzcd} \overset{\alpha}{F} \arrow[r, "f"] \arrow[rr, bend right, "g = T\circ f"'] & \overset{\beta}{V} \arrow[r, "T"] & \overset{\gamma}{W} \end{tikzcd}$$

$$\begin{tikzcd} \underset{\alpha=\{1\}}{F} \arrow[r, "g"] & \overset{\gamma}{W} \end{tikzcd}$$

$$[g]_\alpha^\gamma = \xcancel{\begin{pmatrix} | \\ [g(\vec{v}_j)]_\gamma \\ | \end{pmatrix}} = \begin{pmatrix} | \\ [g(1)]_\gamma \\ | \end{pmatrix}$$

$$
\begin{tikzcd}
\vec{v} \arrow[d, mapsto] & \in & V_\alpha \arrow[rr, "T"] \arrow[d, "\phi_\alpha"] & & W_\beta \arrow[d, "\phi_\beta"'] & \ni & \vec{w} \arrow[d, mapsto] \\
{[\vec{v}]_\alpha} & \in & F^n \arrow[rr, "A"', "{\parallel}" description, "{[T]_\alpha^\beta}"' yshift=-12pt] & & F^m & \ni & {[\vec{w}]_\beta}
\end{tikzcd}
$$

$$[\vec{w}]_\beta = [T(\vec{v})]_\beta = [T]_\alpha^\beta [\vec{v}]_\alpha$$