Lecture 5:

Zorn's Lemma

Let S be a partially ordered set. If every chain of S has an upper bound in S, then S contains a maximal elements.

Definition 1: (Partially ordered) A partially ordered on a (non-empty) set S is a binary relation on S, denoted $\leq$, which satisfies:

- for $\forall s \in S$, $s \leq s$
- if $s \leq s'$ and $s' \leq s$, then $s = s'$
- if $s \leq s'$ and $s' \leq s''$, then $s \leq s''$.

Definition 2: If every elements in a partially ordered set S is comparable under $\leq$, then S is called a totally ordered set.

Definition 3: A chain is a collection of elements in S satisfying:
if $s_1 \in S$ and $s_2 \in S$, then either $s_1 \le s_2$ or $s_2 \le s_1$.

For simplicity, we may consider countable chain $s_1 \le s_2 \le \dots$ in our discussion for the ease of explanation
(not necessarily countable)

Definition 4: A maximal element m of a partially ordered set S is defined as follows: for $\forall s \in S$ to which m is comparable, $s \le m$.

Definition 5 An upper bound $\tilde{C}$ of a chain $\{C_i\}_{i \in I}$ is that for all $i \in I$,
$C_i \le \tilde{C}$.

Zorn's lemma (adapted to 2048)

Let V be a vector space. Let S be the collection of linearly independent subsets of V. Then S is partially ordered under $\subseteq$.

Assume every chain $\{L_\alpha\}_{\alpha \in I}$ of S has a upper bound.

(Obviously, $\bigcup_\alpha L_\alpha$ is an upper bound of the chain. We need to show that

$\bigcup L_\alpha$ is linearly independent)

Then: S has a maximal element.

Theorem: Every vector space has a basis.

Proof: Let $\mathcal{C}$ be the collection of all linearly independent subsets of V.

For any chain $\{S_i\}_{i\in I}$ (we may consider a countable chain $S_1 \subset S_2 \subset \ldots$ for easier interpretation.)

$\bigcup_{i\in I} S_i$ is also linearly independent. $\therefore$ $\bigcup_i S_i \in \mathcal{C}$.

By Zorn's lemma, $\exists$ maximal linearly independent set M.

We claim that $\mathrm{span}(M) = V$.

If not, $\exists\, \vec{v} \in V \ni \vec{v} \notin \mathrm{Span}(M)$.

Then: $M \cup \{\vec{v}\}$ is linearly independent.

But $M \subset M \cup \{\vec{v}\}$. Contradition to Zorn's lemma.

$\therefore$ ① $\mathrm{Span}(M) = V$

② M is L.I.

$\Rightarrow$ M is a basis.

Theorem: Every spanning set of a non-zero vector space V contains a basis of V.

Proof: Let S be a spanning set of V.

Let $\mathcal{C}$ be the collection of linearly independent subsets of S.

Then $\mathcal{C} \neq \phi$ (as $\{\vec{v}\} \in \mathcal{C}$ for any $\vec{v} \in S$, $\vec{v} \neq \vec{0}$)

Then, $\mathcal{C}$ under $\subseteq$ is partially ordered.

Let $\{L_i\}_{i \in I}$ be a chain in $\mathcal{C}$.

Then $\bigcup_i L_i \in \mathcal{C}$ and $\bigcup_i L_i$ is an upper bound.

By Zorn's lemma, there is a maximal element $\mathcal{B}$ in $\mathcal{C}$.

(i.e. a linearly independent subset of S which is maximal)

We'll show that $\mathrm{Span}(\mathcal{B}) = \mathrm{Span}(\mathcal{S}) = V$

It suffices to show that for any $\vec{v} \in \mathcal{S}$, $\vec{v} \in \mathrm{Span}(\mathcal{B})$.

If not, suppose $\vec{v} \notin \mathrm{Span}(\mathcal{B})$.

Then: $\mathcal{B} \cup \{\vec{v}\}$ is linearly independent subset of $\mathcal{S}$.

Hence, $\mathcal{B} \cup \{\vec{v}\} \in \mathcal{C}$. But $\mathcal{B} \cup \{\vec{v}\} \supsetneq \mathcal{B}$.

Contradicting to the fact that $\mathcal{B}$ is maximal.

$\therefore$ $\vec{v} \in \mathrm{Span}(\mathcal{B})$.

$\therefore$ $\mathrm{Span}(\mathcal{B}) = \mathrm{Span}(\mathcal{S}) = V$.

$\therefore$ $\mathcal{B}$ is a basis.

Remark: To find a basis inside a spanning set $\mathcal{S}$, it's natural to find a minimal spanning set of V inside $\mathcal{S}$.

If $\mathcal{M} \subset \mathcal{S}$ is minimal, then $\mathcal{M}$ is linearly independent.

If not, we can find $v \in \mathcal{M} \ni \mathrm{Span}(\mathcal{M} \setminus \{\vec{v}\}) = \mathrm{Span}(\mathcal{M})$, contradicting the minimality of $\mathcal{M}$.

One might consider Zorn's lemma as follows:

Let $\mathcal{C}$ be the set of all spanning subsets of $\mathcal{S}$, partially order $\mathcal{C}$ by reverse inclusion. That is: $S_1 \in \mathcal{C}$ and $S_2 \in \mathcal{C}$, $S_1 \leq S_2$ iff $S_1 \supseteq S_2$.

For any chain $\{S_i\}_{i \in I}$ in $\mathcal{C}$, $\bigcap_{i \in I} S_i$ is the upper bound.

If $\bigcap_{i \in I} S_i \in \mathcal{C}$, then Zorn's lemma tells us $\mathcal{C}$ has a maximal element (i.e. minimal spanning set)

BUT: $\bigcap_{i \in I} \mathcal{S}_i$ MAY NOT always in $\mathcal{C}$!!

Linear Transformation

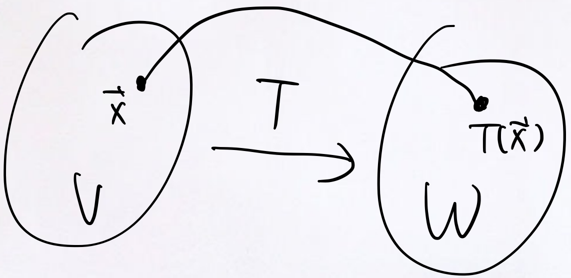
Definition: Let V and W be vector spaces over F.

A linear transformation from V to W is a map $T: V \to W$

such that: (a) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

(b) $T(a\vec{x}) = a\,T(\vec{x})$

for all $\vec{x}, \vec{y} \in V$, $a \in F$.



<u>Proposition:</u> Let $T : V \to W$ be a linear transformation. Then:

(i) $T(\vec{0}_V) = \vec{0}_W$

(ii) $T\left(\sum_{i=1}^{n} a_i \vec{x}_i\right) = \sum_{i=1}^{n} a_i T(\vec{x}_i)$ $\forall \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in V$
$a_1, a_2, \ldots, a_n \in F$.

(T preserves linear combination)

(i) $T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$

$\Rightarrow T(\vec{0}_V) = \vec{0}_W$. (Cancellation law)

(ii) Use math. induction (exercise)

Examples: • For any vector spaces V and W, we have:

(a) The zero transformation $T_0 : V \to W$ defined by $T_0(\vec{x}) := \vec{0}_W$ for $\forall \vec{x} \in V$

(b) The identity transformation $I_V : V \to V$ defined by $I_V(\vec{x}) = \vec{x}$ for $\forall \vec{x} \in V$.

• Let $A \in M_{m\times n}(F)$ be a $m\times n$ matrice . F ,

Define: $L_A : F^n \to F^m$ as: (F^n = space of col vectors of size n)

$$L_A(\vec{x}) \overset{def}{=} A\vec{x}$$

L_A is called the left multiplication by A.

• $T : M_{m\times n}(F) \to M_{n\times m}(F)$ defined by $T(A) \overset{def}{=} A^t$ (transpose of A)

- $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$

  (derivative of f)

  is a lin. transf.

- Let a and $b \in \mathbb{R}$, $a < b$. Then,

  $T : C(\mathbb{R}) \to \mathbb{R}$ defined by :

  (Space of continuous functions)

  $$T(f) \stackrel{def}{=} \int_a^b f(t)\,dt$$

Null space or Range

Definition: Let V and W be vector spaces and $T: V \to W$ be a linear transformation

Then, the null space (or kernel) of T is defined as:

$$N(T) \overset{\text{def}}{:=} \{ \vec{x} \in V : T(\vec{x}) = \vec{0}_W \} \subset V$$

the range (or image) of T is defined as:

$$R(T) := \{ T(\vec{x}) : \vec{x} \in V \} \subset W$$

e.g. For $I_V : V \to V$, (identity) $\quad N(I_V) = \{ \vec{0}_V \}$, $R(I_V) = V$

For $T_0 : V \to W$, (zero transf) $\quad N(T_0) = V$, $R(T_0) = \{ \vec{0}_W \}$

- $L_A : F^n \to F^m \quad (A \in M_{m\times n}(F))$

$N(L_A) = N(A) =$ null space of A

$R(L_A) = \mathcal{C}(A) =$ col space of A (space of linear combination of col vectors of A)

$$A = \begin{pmatrix} | & | & \cdots & | \end{pmatrix}$$

- For $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$, then:

$$N(T) = \{ a_0 \in P_n(\mathbb{R}) : a_0 \in \mathbb{R} \}$$

$$R(T) = P_{n-1}(\mathbb{R})$$