Lecture 4:

Recall:

Quotient Space

Definition: Let V be a vector space over F and let W be a subspace of V. Let $\vec{v} \in V$. Define:

$$\vec{v} + W = \{ \vec{v} + \vec{w} : \vec{w} \in W \}$$

$\vec{v} + W$ is called a coset of W in V.

Remark: $\vec{v} \in \vec{v} + W$.

Definition: The set V/W (called V mod W), is the set defined by

$$V/W = \{ \vec{v} + W : \vec{v} \in V \}$$

(collection of cosets of W in V)

Proposition: Let $\vec{v}, \vec{v}' \in V$. Then: $\vec{v} + W = \vec{v}' + W$ iff $\vec{v} - \vec{v}' \in W$.

Proof: $(\Rightarrow)$ Let $\vec{v} + W = \vec{v}' + W$.

$\because$ $\vec{v} \in \vec{v} + W = \vec{v}' + W$. $\therefore$ $\vec{v} = \vec{v}' + \vec{w}$ for some $\vec{w} \in W$

$\therefore$ $\vec{v} - \vec{v}' = \vec{w} \in W$.

$(\Leftarrow)$ Suppose $\vec{v} - \vec{v}' \in W$.

Let $\vec{w} = \vec{v} - \vec{v}'$. Then: $v = \vec{v}' + w$, for some $\vec{w} \in W$.

$\therefore$ $\vec{v} + W \subset \vec{v}' + W$. Similarly, $\vec{v}' = \vec{v} + \vec{w}'$ for some $\vec{w}' \in W$.

$\Rightarrow \vec{v}' + W \subset \vec{v} + W$.

Definition: Define:

$$(\vec{v} + W) + (\vec{v}' + W) \overset{\text{def}}{:=} (\vec{v} + \vec{v}') + W \quad \text{(addition)}$$

$$a \cdot (\vec{v} + W) \overset{\text{def}}{:=} a\vec{v} + W \quad \text{(Scalar multiplication)}$$

Proposition: Suppose $\vec{v} + W = \vec{v}' + W$. Then: for any $\vec{v}'' + W \in V/W$.

- $(\vec{v} + W) + (\vec{v}'' + W) = (\vec{v}' + W) + (\vec{v}'' + W)$
- $a \cdot (\vec{v} + W) = a \cdot (\vec{v}' + W)$ for any $a \in F$.

Proof: Homework!

Remark: Addition and scalar multiplication are well-defined.

Theorem: With addition and scalar multiplication defined above, V/W is a vector space over F, called the <u>quotient space</u>.

Proof: Homework!

Examples of quotient space

- Let $W = \{\vec{0}\}$. V/W is the (isomorphic) same as V.

  Let $W = V$. V/V is the same as $\{\vec{0}\}$.

$$\vec{v} + W = \vec{v}\,' + W \text{ iff } \vec{v} - \vec{v}\,' \in W = \{\vec{0}\}$$
$$\text{iff } \vec{v} - \vec{v}\,' = \vec{0}$$
$$\text{iff } \vec{v} = \vec{v}\,'$$

- Let $V = \mathbb{R}^2$. Let W be the y-axis.

Recall: $(x, y) + W = (x', y') + W$ iff $(x, y) - (x', y') \in W$

iff $x - x' = 0$ or $x = x'$

$\therefore$ a vector in V/W is determined by the x-coordinate.

- Let $V = F^\infty$ (infinite sequence)

Let $W :\overset{def}{=} \{(0, x_2, x_3, \dots) : x_i \in F\}$.

As above, two vectors in V/W are the same iff they have the same first coordinate.

$(x_1, x_2, \quad) + W = (x_1', x_2', \quad) + W$ iff $(x_1 - x_1', x_2 - x_2', \quad) \in W$

iff $x_1 - x_1' = 0$ iff $x_1 = x_1'$

$\therefore$ V/W is the same (isomorphic) as F

<u>Remark:</u> Even V and W are infinite dimensional,

V/W is one-dimensional!

Proposition: Suppose V is finite-dimensional. Then:

$$\dim(V/W) = \dim(V) - \dim(W).$$

Proof: Let $\{\vec{w}_1, \ldots, \vec{w}_n\}$ be a basis of W.

Extend it to a basis $\{\vec{w}_1, \ldots, \vec{w}_n, \vec{v}_1, \ldots, \vec{v}_k\}$ of V.

Then: $\dim(W) = n$, $\dim(V) = n + k$

We'll prove that $\{\vec{v}_1 + W, \ldots, \vec{v}_k + W\}$ forms a basis of V/W.

If so, we'll have: $\dim(V/W) = k = \underbrace{(n+k)}_{\dim(V)} - \underbrace{n}_{\dim(W)}$

Linear independence:

Suppose: $a_1(\vec{v}_1 + W) + \ldots + a_k(\vec{v}_k + W) = \vec{0} + W$

$$\Rightarrow (a_1\vec{v}_1 + \ldots + a_k\vec{v}_k) + W = \vec{0} + W$$

$\therefore \quad a_1\vec{v}_1 + \dots + a_k\vec{v}_k \in W$

$\Rightarrow \quad a_1\vec{v}_1 + \dots + a_k\vec{v}_k = b_1\vec{w}_1 + \dots + b_n\vec{w}_n$ for some $b_1, \dots, b_n \in F$.

$\Rightarrow \quad a_1\vec{v}_1 + \dots + a_k\vec{v}_k - b_1\vec{w}_1 - \dots - b_n\vec{w}_n = \vec{0}$

As $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_n\}$ is linearly independent,

$a_1 = \dots = a_k = 0$ and $b_1 = \dots = b_n = 0$.

$\therefore \quad \{\vec{v}_1 + W, \dots, \vec{v}_k + W\}$ is linear independent.

<u>Span:</u> Let $\vec{v} + W \in V/W$.

Then: $\vec{v} = a_1\vec{w}_1 + \dots + a_n\vec{w}_n + b_1\vec{v}_1 + \dots + b_k\vec{v}_k$ for some a_i's and b_j's.

$\Rightarrow \vec{v} + W = \Big(b_1\vec{v}_1 + \dots + b_k\vec{v}_k + \underbrace{a_1\vec{w}_1 + \dots + a_n\vec{w}_n}_{\in W}\Big) + W$

$= b_1(\vec{v}_1 + W) + \dots + b_k(\vec{v}_k + W)$

Existence of basis

For a finite-dimensional vector space, the basis can be constructed as follows:

$\vec{v}_1 \neq \vec{0}$

$\{\vec{v}_1\}$ ← Linear independent

↓

$\{\vec{v}_1, \vec{v}_2\}$ ← Attach one more vector $\vec{v}_2 \ni \{\vec{v}_1, \vec{v}_2\}$ is linearly independent

↓

⋮

$\{\vec{v}_1, \ldots, \vec{v}_n\}$ ← must stop at some point.

Constructive proof for the existence of basis.

Example: Consider $F^\infty = \{(a_1, a_2, \dots) : a_j \in F\}$.

Let $S_i = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_i\}$

$\vec{e}_1 = (1, 0, \dots 0)$, $\vec{e}_2 = (0, 1, 0, \dots)$

Then: $S_1 \subset S_2 \subset \dots \subset S_i \subset \dots$

Let $S = \bigcup_i S_i$, which is linearly independent.

Obviously $\text{span}(S) \neq F^\infty$.

So, we can find $\vec{v} \notin \text{span}(S) \ni S \cup \{\vec{v}\}$ is linearly independent

We can repeat the process.

Question: will the process stop??