Lecture 3:

Recall:

Direct Sum:

- Let U and W be subspaces of V. Then: V is said to be the direct sum of U and W, denoted by $V = U \oplus W$, if $V = U + W$ and $U \cap V = \{\vec{0}\}$.
- $V = U \oplus W$ iff for $\forall \vec{v} \in V$, $\exists !$ vectors $\vec{u} \in U$ and $\vec{w} \in W$ such that $\vec{v} = \vec{u} + \vec{w}$
- Let $V = U \oplus W$. Define: $P : V \to U$ as follows:

For any $\vec{v} \in V$, write $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u} \in U$ and $\vec{w} \in W$.

Then: define: $P(\vec{v}) = \vec{u}$

- V is said to be a direct sum of subspaces $U_1, U_2, \ldots, U_k$, denoted as $V = U_1 \oplus U_2 \oplus \ldots \oplus U_k$, if for $\forall \vec{v} \in V$, $\exists!$ vectors $\vec{u}_i \in U_i$ $(1 \leq i \leq k)$ $\ni$ $\vec{v} = \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_k$.

- $U_1 \oplus \ldots \oplus U_k = (\cdots ((U_1 \oplus U_2) \oplus U_3) \oplus \ldots \oplus U_k)$

- $V = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ iff :

  1. $V = U_1 + U_2 + \ldots + U_k$
  2. $U_r \cap \sum_{i \neq r} U_i = \{\vec{0}\}$ for $1 \leq r \leq k$.

Dimension of direct sum

Theorem: Let V be a finite-dim vector space. $U_1, U_2, \ldots, U_m$ are subspaces of V. Then:

$$\dim(U_1 \oplus U_2 \oplus \ldots \oplus U_m) = \sum_{i=1}^{m} \dim(U_i)$$

Proof: Let β_i = basis of U_i for $i = 1, 2, \ldots, m$.

Let $\beta = \beta_1 \dot\cup \beta_2 \dot\cup \ldots \dot\cup \beta_m$ (disjoint union)

For $\forall \vec{v} \in U_1 \oplus \ldots \oplus U_m$, $\exists!\ \vec{u}_1 \in U_1, \vec{u}_2 \in U_2, \ldots, \vec{u}_m \in U_m \Rightarrow$

$$\vec{v} = \vec{u}_1 + \ldots + \vec{u}_m.$$

Each $\vec{u}_i$ can be written as a linear combination of elements in β_i.

$\therefore$ $\mathrm{Span}(\beta) = U_1 \oplus \ldots \oplus U_m$

β is linear independent.

Let $\vec{0} = (a_1^1 \underset{\substack{\uparrow \\ \beta_1}}{u_1^1} + a_2^1 \underset{\substack{\uparrow \\ \beta_1}}{u_2^1} + \dots + a_n^1 \underset{\substack{\uparrow \\ \beta_1}}{u_n^1}) + \dots + (a_1^m \underset{\substack{\uparrow \\ \beta_m}}{u_1^m} + \dots + a_n^m \underset{\substack{\uparrow \\ \beta_m}}{u_n^m})$

Then: each $a_1^j u_1^j + \dots + a_n^j u_n^j = \vec{0}$ for $\forall j$

$\Rightarrow a_1^j = a_2^j = \dots = a_n^j = 0$ for $\forall j$.

$\therefore$ β is linear independent.

$\therefore$ β is a basis.

$\therefore \dim(U_1 \oplus \dots \oplus U_m) = |\beta| = \sum_{i=1}^{m} \dim(U_i)$.

<u>Remark:</u> In general,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(Homework!)

Direct product space

Definition: Let V_1 and V_2 are vector spaces over F. Define:

$$V_1 \times V_2 = \{(\vec{x}, \vec{y}) : \vec{x} \in V_1 \text{ and } \vec{y} \in V_2\}$$

(called the direct product of V_1 and V_2)

Define:
- $(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2) = (\vec{x}_1 + \vec{x}_2, \vec{y}_1 + \vec{y}_2)$ for $\forall \vec{x}_1, \vec{x}_2 \in V_1$, $\vec{y}_1, \vec{y}_2 \in V_2$
- $a(\vec{x}, \vec{y}) = (a\vec{x}, a\vec{y})$ for $\forall a \in F$, $\vec{x} \in V_1$, $\vec{y} \in V_2$.

Then: $V_1 \times V_2$ forms a vector space over F.

Exercise: Check.

Theorem: $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$

Proof: (Idea) Let $\beta_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ = basis for V_1.

$\beta_2 = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ = basis for V_2.

Then: $\{(\vec{v}_1, \vec{0}_2), \ldots, (\vec{v}_n, \vec{0}_2), (\vec{0}_1, \vec{w}_1), \ldots, (\vec{0}_1, \vec{w}_m)\}$ forms a basis for $V_1 \times V_2$ (where $\vec{0}_1$ is the zero in V_1

$\vec{0}_2$ is the zero in V_2]

[Check! Exercise)

Remark: • For finite-dimensional vector space, direct product can be considered as direct sum.

Consider $X = Y_1 \times Y_2$ where $\dim(Y_1) < \infty$, $\dim(Y_2) < \infty$.

Let
$$X_1 = \{(\vec{y}_1, \vec{0}_2) : \vec{y}_1 \in Y_1\}$$
$$X_2 = \{(\vec{0}_1, \vec{y}_2) : \vec{y}_2 \in Y_2\}$$
subspaces of X.

(where $\vec{0}_1$ = zero vector in Y_1
$\vec{0}_2$ = zero vector in Y_2)

Then: $X = X_1 \oplus X_2$

$= Y_1 \times Y_2$

Remark: • Consider $X = \mathbb{R}^\infty = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$

Then: $\dim(X) = \infty$.

Consider: $X_1 = \{(a_1, 0, 0, \dots,) : a_1 \in \mathbb{R}\}$;

$X_2 = \{(0, a_2, 0, \dots) : a_2 \in \mathbb{R}\}$;

$\vdots$

$X_i = \{(0, 0, \dots, a_i, 0 \dots 0) : a_i \in \mathbb{R}\}$

$\vdots$

Define: $X_1 \oplus X_2 \oplus X_3 \oplus \dots = \{\vec{x} = \vec{x}_{i_1} + \vec{x}_{i_2} + \dots + \vec{x}_{i_k} : \vec{x}_{i_j} \in X_{i_j},\ k \in \mathbb{N}\}$.

Then:

$X_1 \oplus X_2 \oplus \dots = \{(a_1, \dots, a_k, 0, 0, \dots) : a_i \in \mathbb{R},\ k \in \mathbb{N}\} \subsetneq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots$

Direct product $\neq$ Direct Sum

- Direct product = collection of infinite sequence
- Direct sum = collection of finite sum / finite sequence.

Quotient Space

Definition: Let V be a vector space over F and let W be a subspace of V. Let $\vec{v} \in V$. Define:

$$\vec{v} + W = \{\vec{v} + \vec{w} : \vec{w} \in W\}$$

$\vec{v} + W$ is called a coset of W in V.

Remark: $\vec{v} \in \vec{v} + W$.

Definition: The set V/W (called V mod W), is the set defined by $V/W = \{\vec{v} + W : \vec{v} \in V\}$

(collection of cosets of W in V)