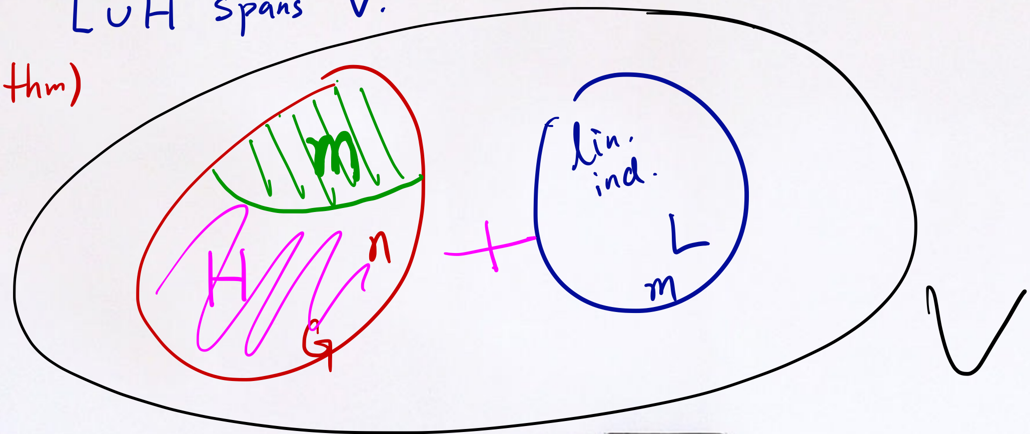
Lecture 2

Theorem: Let V be a vector space.
Let $G \subset V$ be a spanning set for V consisting of n vectors.
and $L \subset V$ be a linearly independent subset consisting of m vectors.
Then, $m \leq n$ and $\exists\, H \subset G$ consisting of exactly $n-m$ vectors
such that $L \cup H$ spans V.

(Replacement thm)



Dimension

Cor 1: Let V be a vector space having a finite basis. Then, every basis of V contains the same number of vectors.

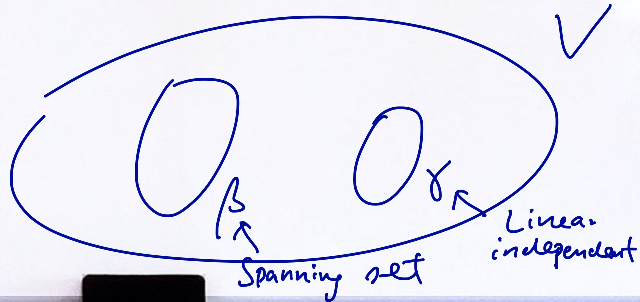
Pf: Let β and γ be two bases of V.

Since β spans V and γ is lin. independent,

then $|\gamma| \leq |\beta|$ (by replacement Thm)

Similarly, $|\beta| \leq |\gamma|$

$\Rightarrow |\gamma| = |\beta|$.



Definition: A vector space V is called <u>finite-dimensional</u> if it has a finite basis. The <u>dimension</u> of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-dimensional is called <u>infinite-dimensional</u>

Example:

- F^n is n-dimensional
- $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional

Direct Sum: Let $\mathcal{U}$ and W be subspaces of V.

Then: $\mathcal{U} + W = \{ \vec{x} + \vec{y} : \vec{x} \in \mathcal{U} \text{ and } \vec{y} \in W \}$ is also a subspace of V (Check!)

Definition: V is said to be the **direct sum** of $\mathcal{U}$ and W, denoted by $V = \mathcal{U} \oplus W$ if $V = \mathcal{U} + W$ and $\mathcal{U} \cap V = \{\vec{0}\}$.

Lemma: $V = \mathcal{U} \oplus W$ iff for $\forall \vec{v} \in V$, $\exists !$ vectors $\vec{u} \in \mathcal{U}$ and $\vec{w} \in W \ni \vec{v} = \vec{u} + \vec{w}$

Proof: $(\Rightarrow)$ If $\vec{v} \in V$, then $\vec{v} = \vec{u} + \vec{w}$ for some $\vec{u} \in \mathcal{U}$ and $\vec{w} \in W$ $(\because V = \mathcal{U} \oplus W)$

For uniqueness, let $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2$.

Then: $\vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1 \in \mathcal{U} \cap W = \{\vec{0}\}$. $\therefore$

$\vec{u}_1 - \vec{u}_2 = \vec{0} \Rightarrow \vec{u}_1 = \vec{u}_2$

$\vec{w}_2 - \vec{w}_1 = \vec{0} \Rightarrow \vec{w}_1 = \vec{w}_2$

($\Leftarrow$) $V = U + W$ is obvious.

Now, let $\vec{z} \in U \cap W$. $\exists!\ \vec{u}$ and $\vec{w} \ni \vec{z} = \vec{u} + \vec{w}$.

Then: $$\vec{z} = \underset{\in U}{\vec{z}} + \underset{\in W}{\vec{0}} = \underset{\in U}{\vec{0}} + \underset{\in W}{\vec{z}} \Rightarrow \vec{u} = \vec{0} \text{ and } \vec{w} = \vec{0}$$

$$\Rightarrow \vec{z} = \vec{0}.$$

$$\therefore\ U \cap W = \{\vec{0}\}$$

Projection operators

Definition: Suppose $V = U \oplus W$. Define : $P : V \to U$ as follows :

For any $\vec{v} \in V$, write $\vec{v} = \vec{u} + \vec{w}$ where $\vec{u} \in U$ and $\vec{w} \in W$.

Then: define : $P(\vec{v}) = \vec{u}$

Remark:

1. P is well-defined
2. $P \circ P = P$

Definition V is said to be a direct sum of subspaces $U_1, U_2, \ldots, U_k$, denoted as $V = U_1 \oplus U_2 \oplus \ldots \oplus U_k$, if for $\forall \vec{v} \in V$, $\exists!$ vectors $\vec{u}_i \in U_i$ $(1 \leq i \leq k)$ $\ni$ $\vec{v} = \vec{u}_1 + \vec{u}_2 + \ldots + \vec{u}_k$.

Remark: • $U_1 \oplus \ldots \oplus U_k = (\cdots((U_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_k)$

• $V = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ iff :

1. $V = U_1 + U_2 + \ldots + U_k$
2. $U_r \cap \sum_{i \neq r} U_i = \{\vec{0}\}$ for $1 \leq r \leq k$.

Dimension of direct sum

Theorem: Let V be a finite-dim vector space. $U_1, U_2, \ldots, U_m$ are subspaces of V. Then:

$$\dim(U_1 \oplus U_2 \oplus \ldots \oplus U_m) = \sum_{i=1}^{m} \dim(U_i)$$