Lecture 16: Recall:

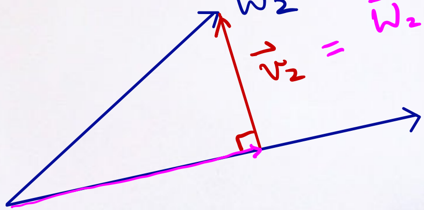
Prop: Let V be an inner product space and $S_n = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be a linearly independent subset of V. Define:

$$S_n' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \text{ where } \vec{v}_1 = \vec{w}_1 \text{ and}$$

for $k = 2, \ldots, n$,

$$\vec{v}_k \stackrel{\text{def}}{=} \vec{w}_k - \sum_{j=1}^{k-1} \left(\frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \right) \vec{v}_j$$

Then: S_n' is orthogonal and $\text{Span}(S_n') = \text{span}(S_n)$



$\vec{v}_2 = \vec{w}_2 - c_1 \vec{v}_1$ where $c_1 = \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}$

<u>Proof:</u> We prove by induction on n.

For $n=1$, we simply have $S_1' = S_1$. $\therefore$ The statement is obviously true.

Suppose the statement is true for $n = m-1$.

That's, $S'_{m-1} = \{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ is orthogonal and

$$\operatorname{Span}(S'_{m-1}) = \operatorname{Span}(S_{m-1})$$

Induction hypothesis

Now, consider a lin. independent subset $S_m = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_{m-1}, \vec{w}_m\}$

Then: for $\vec{v}_m \stackrel{\text{def}}{=} \vec{w}_m - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j$, we have:

$$\langle \vec{v}_m, \vec{v}_i \rangle = \langle \vec{w}_m, \vec{v}_i \rangle - \sum_{j=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \langle \vec{v}_j, \vec{v}_i \rangle \quad \text{for } i = 1, \dots, m-1$$

$$= \cancel{\langle \vec{w}_m, \vec{v}_i \rangle} - \cancel{\frac{\langle \vec{w}_m, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \langle \vec{v}_i, \vec{v}_i \rangle} = 0$$

$\therefore$ S_m' is orthogonal.

Also, $\vec{v}_m \neq \vec{0}$ since otherwise, $\vec{w}_m \in \mathrm{Span}(S_{m-1}')$

$= \mathrm{Span}(S_{m-1})$

$= \mathrm{Span}(\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{m-1}\})$

contradicting the condition that S_m is linearly independent.

Hence, $S_m' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m-1}, \vec{v}_m\}$ is orthogonal subset consisting of non-zero vectors. $\therefore$ S_m' is linearly independent.

Also, $\underset{\dim = m}{\mathrm{Span}(S_m')} \subset \underset{\dim = m}{\mathrm{Span}(S_m)}$ $\Rightarrow$ $\mathrm{Span}(S_m') = \mathrm{Span}(S_m)$

The above construction of an orthogonal basis is called <u>Gram-Schmidt process.</u>

Consider $V = P(\mathbb{R})$ equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt .$$

Let $\beta = \{1, x, x^2, \ldots, x^n, \ldots\}$ be standard ordered basis for $P(\mathbb{R})$.

Take $\vec{v}_1 = 1$.

Then: $$\vec{v}_2 = x - \frac{\overset{=0}{\langle x, \vec{v}_1\rangle}}{\underset{=2}{\|\vec{v}_1\|^2}}\,\overset{=1}{\vec{v}_1} = x$$

$$\vec{v}_3 = x^2 - \frac{\overset{=2/3}{\langle x^2, \vec{v}_1\rangle}}{\underset{=2}{\|\vec{v}_1\|^2}}\,\vec{v}_1 - \frac{\overset{=0}{\langle x^2, \vec{v}_2\rangle}}{\|\vec{v}_2\|^2}\,\vec{v}_2 = x^2 - \frac{1}{3}$$

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$$\vec{v}_4 = x^3 - \cancel{\frac{\langle x^3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}} \vec{v}_1 \boxed{- \frac{\langle x^3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2} - \cancel{\frac{\langle x^3, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}} \vec{v}_3$$

$$= x^3 - \frac{3}{5} x \quad \text{and so on, ...}$$

for $P(\mathbb{R})$

This produces an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots\}$, whose elements are called Legendre polynomial.

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Corollary: Let V be a non-zero finite-dim inner product space. Then, V has an orthonormal basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ s.t. $\forall \vec{x} \in V$, we have: $\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i$

Corollary: Let V be a non-zero finite-dim inner product space with an orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Let T be a linear operator on V. Let $A = [T]_\beta$. Then: $A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$.

Proof:

$$[T]_\beta = \begin{pmatrix} \cdots & \begin{array}{c} | \\ [T(\vec{v}_j)]_\beta \\ | \end{array} & \cdots \end{pmatrix} \qquad T(\vec{v}_j) = \sum_{i=1}^{n} \underbrace{\langle T(\vec{v}_j), \vec{v}_i \rangle}_{= A_{ij}} \vec{v}_i$$

Orthogonal complement

Def: Let S be a non-empty subset of an inner product space V. The orthogonal complement of S is defined as:

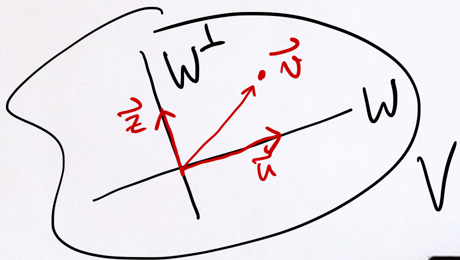
$$S^{\perp} \overset{\text{def}}{=} \{ \vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for } \forall \vec{y} \in S \}$$

Proposition: Let V be an inner product space and $W \subset V$ a finite-dim subspace of V. Then: $\forall \vec{y} \in V$, $\exists ! \, \vec{u} \in W$ and $\vec{z} \in W^{\perp}$ such that $\vec{y} = \vec{u} + \vec{z}$.

Furthermore, if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an orthonormal basis for W, then:

$$\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

The vector $\vec{u} \in W$ is called the orthogonal projection of $\vec{y}$ on W.



Proof: Given $\vec{y} \in V$, we set $\vec{u} \overset{\text{def}}{=} \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i \in W$

and $\vec{z} \overset{\text{def}}{=} \vec{y} - \vec{u}$. Then: $\vec{y} = \underbrace{\vec{u}}_{W} + \underbrace{\vec{z}}_{W^{\perp}\,??}$.

Now, $\langle \vec{z}, \vec{v}_j \rangle = \langle \vec{y} - \vec{u}, \vec{v}_j \rangle = \langle \vec{y}, \vec{v}_j \rangle - \langle \vec{u}, \vec{v}_j \rangle$

$$= \langle \vec{y}, \vec{v}_j \rangle - \underbrace{\sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \langle \vec{v}_i, \vec{v}_j \rangle}_{= \langle \vec{y}, \vec{v}_j \rangle}$$

$$\therefore \langle \vec{z}, \overbrace{\sum_{i=1}^{k} b_k \vec{v}_i}^{\in W} \rangle = \sum_{i=1}^{k} \overline{b_k} \overbrace{\langle \vec{z}, \vec{v}_i \rangle}^{0 = 0} = 0$$

$$\therefore \vec{z} \in W^{\perp}$$

For uniqueness, suppose $\exists\, \vec{u}\,' \in W$ and $\vec{z}\,' \in W^{\perp}$ such that:

$$\vec{y} = \underset{\substack{\uparrow \\ W}}{\vec{u}} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} = \underset{\substack{\uparrow \\ W}}{\vec{u}\,'} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}\,'} \implies \underset{\substack{\uparrow \\ W}}{\vec{u} - \vec{u}\,'} = \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}\,' - \vec{z}} \in W \cap W^{\perp}$$

Claim: $W \cap W^{\perp} = \{\vec{0}\}$

Pf: Take $\vec{w} \in W \cap W^{\perp}$. Then: $\langle \underset{\substack{\uparrow \\ W}}{\vec{w}}, \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{w}} \rangle = 0$

$$\iff \vec{w} = \vec{0}$$

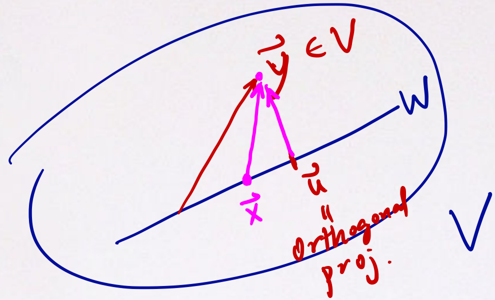
This implies: $\vec{u} - \vec{u}\,' = \vec{z}\,' - \vec{z} = \vec{0}$

$$\iff \vec{u} = \vec{u}\,' \text{ and } \vec{z} = \vec{z}\,'.$$

Corollary: With notations as above, then:

$$\|\underset{\overset{\uparrow}{V}}{\vec{y}} - \underset{\overset{\uparrow}{W}}{\vec{x}}\| \geq \|\underset{\overset{\uparrow}{V}}{\vec{y}} - \underset{\overset{\uparrow}{W}}{\vec{u}}\| \quad \text{for } \forall \vec{x} \in W$$

and equality holds iff $\vec{x} = \vec{u}$



Remark: Orthogonal projection is the vector in W closest to $\vec{y}$.

Proof: Let $\vec{x} \in W$. Then: $\vec{y} = \underset{W}{\vec{u}} + \underset{W^{\perp}}{\vec{z}} \Rightarrow \vec{z} = \vec{y} - \vec{u}$

$$\|\vec{y} - \vec{x}\|^2 = \|\vec{u} + \vec{z} - \vec{x}\|^2 = \langle (\underset{W}{\vec{u}} - \underset{W}{\vec{x}}) + \underset{W^{\perp}}{\vec{z}}, (\underset{W}{\vec{u}} - \underset{W}{\vec{x}}) + \underset{W^{\perp}}{\vec{z}} \rangle$$

$$= \langle \vec{u} - \vec{x}, \vec{u} - \vec{x} \rangle + \underset{=0}{\langle \underset{W}{\vec{u} - \vec{x}}, \underset{W^{\perp}}{\vec{z}} \rangle} + \underset{=0}{\langle \vec{z}, \vec{u} - \vec{x} \rangle} + \langle \vec{z}, \vec{z} \rangle$$

$$= \|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 \geq \|\vec{z}\|^2 = \|\vec{y} - \vec{u}\|^2$$

The equality holds iff $\|\vec{u} - \vec{x}\|^2 = 0$ iff $\vec{u} = \vec{x}$.

Proposition: Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can extended to an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

(b) If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$

(c) If W is any subspace of V, then:

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Proof: (a) We first extend S to a basis:

$$\{\overbrace{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{L. I.}}}^{S}, \vec{w}_{k+1}, \ldots, \vec{w}_n\} \text{ for } V.$$

Then, we apply the G-S process to this basis.

$\because$ S is orthonormal, $\therefore$ $\vec{v}_1, \ldots, \vec{v}_k$ remains the same during the G-S process.

So, this process gives an orthonormal basis for V of the form $\{\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k}_{\text{unchanged}}, \underbrace{\vec{v}_{k+1}, \ldots, \vec{v}_n}_{\text{new}}\}$

(b) Exercise. Consider $\vec{w} \in W^\perp$ and write $\vec{w} = \sum_{i=1}^{n} b_i \vec{v}_i$. Use: $\langle \vec{w}, \vec{v}_i \rangle = 0$ for $i=1,2,\ldots,k$

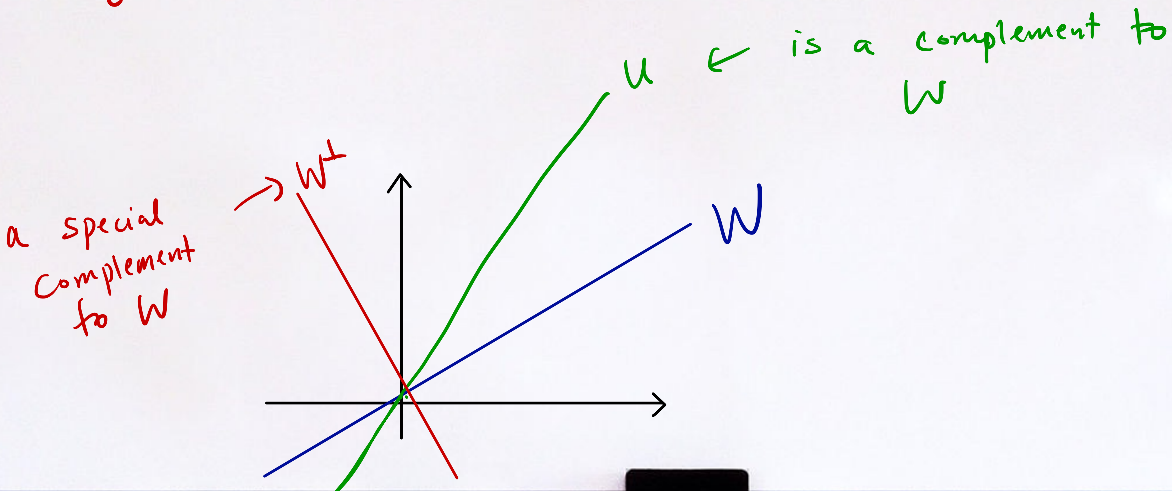
(c) For any W, choose an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for W and extend it to an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

Then:

$$\dim(V) = n = k + (n-k)$$
$$= \dim(W) + \dim(W^\perp)$$

Remark: In fact, a "complement" to a subspace $W \subset V$ $(\dim(V) < \infty)$ is another subspace $U \subset V$ s.t.

$$\begin{cases} \cdot\ W \cap U = \{\vec{0}\} \\ \cdot\ \dim(W) + \dim(U) = \dim(V) \end{cases}$$



Adjoint of a linear operator

Prop: Let V be a finite-dim. inner product space over F.

Then for any linear transformation $g : V \to F$ (linear functional),

$\exists !\, \vec{y} \in V$ s.t. $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Theorem: Let V be a finite-dim inner product space. Let T be a linear operator on V. Then: $\exists !$ linear operator $T^* : V \to V$ such that: $\langle T(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T^*(\vec{y}) \rangle$ for $\forall \vec{x}, \vec{y} \in V$.

T^* is called the adjoint of T.

Proof: Given any $\vec{y} \in V$, the map $g_{\vec{y}} : V \to F$ defined by:

$$g_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y} \rangle$$

is linear ($\because$ $\langle \cdot, \cdot \rangle$ is linear in the 1st argument)

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T is linear

By the previous proposition, $\exists ! \ \vec{y}\,' \in V$ such that

$$\underbrace{\langle T(\vec{x}), \vec{y} \rangle}_{=\, g_{\vec{y}}(\vec{x})} = \langle \vec{x}, \vec{y}\,' \rangle \quad \text{for all } \vec{x} \in V.$$

Now, we uniquely define: $T^* : V \to V$ by $T^*(\vec{y}) = \vec{y}\,'$.

To see that T^* is linear, let $\vec{y}_1, \vec{y}_2 \in V$ and $c \in F$.

Then $\forall \vec{x} \in V$, we have:

$$
\begin{aligned}
\langle \vec{x}, T^*(c\vec{y}_1 + \vec{y}_2) \rangle &= \langle T(\vec{x}), c\vec{y}_1 + \vec{y}_2 \rangle \\
&= \bar{c} \langle T(\vec{x}), \vec{y}_1 \rangle + \langle T(\vec{x}), \vec{y}_2 \rangle \\
&= \bar{c} \langle \vec{x}, T^*(\vec{y}_1) \rangle + \langle \vec{x}, T^*(\vec{y}_2) \rangle \\
&= \langle \vec{x}, c\, T^*(\vec{y}_1) + T^*(\vec{y}_2) \rangle
\end{aligned}
$$

$$\Rightarrow \quad T^*(c\vec{y}_1 + \vec{y}_2) = c\, T^*(\vec{y}_1) + T^*(\vec{y}_2)$$