Lecture 15: Recall:

Inner product and norm

Assume $F = \mathbb{R}$ or $\mathbb{C}$.

Definition: Let V be a vector space over F. An inner product on V is a map $\langle \cdot , \cdot \rangle : V \times V \to F$ s.t. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $c \in F$, it satisfies:

(a) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(b) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$

(c) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$

(d) $\underbrace{\langle \vec{x}, \vec{x} \rangle}_{\in \mathbb{R}} > 0$ if $\vec{x} \neq \vec{0}$

- Let $V = C([0,1])$ be vector space of real-valued continuous functions on $[0,1]$. Then: for $f, g \in V$,

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 f(t)\,g(t)\,dt$$ defines an inner product on V.

$(F = \mathbb{R}, \mathbb{C})$

- Let $V = M_{n \times n}(F)$. For $A, B \in V$, we define:

sum of diagonal entries.

$$\langle A, B \rangle \stackrel{\text{def}}{=} \mathrm{tr}(B^* A)$$

where B^* is the conjugate transpose of B defined by:

$$B^* = \overline{B}^T$$

For $A, B, C \in V$ and $\lambda \in F$, we check:

(a) $$\langle A+B, C\rangle = tr(C^*(A+B)) = tr(C^*A + C^*B)$$
$$= tr(C^*A) + tr(C^*B)$$
$$= \langle A, C\rangle + \langle B, C\rangle$$

(b) $$\langle \lambda A, B\rangle = tr(B^*(\lambda A)) = tr(\lambda(B^*A))$$
$$= \lambda\, tr(B^*A)$$
$$= \lambda \langle A, B\rangle$$

(c) $$\overline{\langle A, B\rangle} = \overline{tr(B^*A)} = tr(\overline{\underbrace{B^*}_{=\overline{B^T}}A}) = tr(B^T\bar{A})$$
$$= tr((B^T\bar{A})^T) = tr(\bar{A}^T \underbrace{(B^T)^T}_{=B})$$
$$= tr(A^*B) = \langle B, A\rangle.$$

$$Tr(C) = Tr(C^T)$$

(d) $\langle A, A \rangle = tr(A^* A) = \sum_{i=1}^{n} (A^* A)_{ii}$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \underbrace{(A^*)_{ik}}_{= \overline{A}^T} A_{ki} \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \underbrace{\overline{A}_{ki} A_{ki}}_{= |A_{ki}|^2}$$

$$\langle A, A \rangle = \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2 \geq 0$$

and $\langle A, A \rangle = 0$ iff $A_{ki} = 0 \quad \forall\, k, i$ (i.e. $A = 0$)

Definition: A vector space V equipped with an inner product is called an inner product space.

If $F = \mathbb{C}$, we call V a complex inner product space.

If $F = \mathbb{R}$, we call V a real inner product space.

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Proposition: Let V be an inner product space. Then, $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall c \in F$, we have:

(a) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b) $\langle \vec{x}, c\vec{y} \rangle = \bar{c} \langle \vec{x}, \vec{y} \rangle$

(c) $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$

(d) $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for $\forall \vec{x} \in V$, then $\vec{y} = \vec{z}$.

<u>Proof:</u> (a) $<\vec{x}, \vec{y}+\vec{z}> = \overline{<\vec{y}+\vec{z}, \vec{x}>}$

$= \overline{<\vec{y}, \vec{x}> + <\vec{z}, \vec{x}>}$

$= \overline{<\vec{y}, \vec{x}>} + \overline{<\vec{z}, \vec{x}>} = <\vec{x}, \vec{y}> + <\vec{x}, \vec{z}>$

(b) $<\vec{x}, c\vec{y}> = \overline{<c\vec{y}, \vec{x}>} = \overline{c<\vec{y}, \vec{x}>} = \bar{c}\ \overline{<\vec{y}, \vec{x}>}$

$= \bar{c} <\vec{x}, \vec{y}>$

(c) $<\vec{x}, \vec{0}> = <\vec{x}, \vec{0}+\vec{0}> = <\vec{x}, \vec{0}> + <\vec{x}, \vec{0}>$

So, $<\vec{x}, \vec{0}> = 0$. Similarly, $<\vec{0}, \vec{x}> = 0$

(d) If $\vec{x} = \vec{0}$, then $<\vec{x}, \vec{x}> = 0$ by (c)

If $\vec{x} \neq \vec{0}$, then $<\vec{x}, \vec{x}> > 0$ by definition.

(e) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for all $\vec{x} \in V$.

then $\langle \vec{x}, \vec{y} - \vec{z} \rangle = 0 \quad \forall \vec{x} \in V$.

In particular, we can choose $\vec{x} = \vec{y} - \vec{z}$.

Then: $\langle \vec{y} - \vec{z}, \vec{y} - \vec{z} \rangle = 0 \Rightarrow \vec{y} - \vec{z} = \vec{0}$ (by (d))

$\Rightarrow \vec{y} = \vec{z}$.

<u>Remark:</u> (a) + (b) together say that the inner product is <u>conjugate linear</u> in the second argument.

Definition: Let V be an inner product space. For $\vec{x} \in V$, we can define the <u>length</u> or <u>norm</u> of $\vec{x}$ by:

$$\|\vec{x}\| \overset{\text{def}}{:=} \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

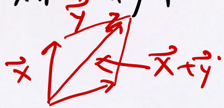
Proposition: Let V be an inner product space over F. Then, $\forall \vec{x}, \vec{y} \in V$ and $\forall c \in F$, we have:

(a) $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

(b) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.

(c) $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$ (Cauchy-Schwarz Inequality)

(d) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (Triangle inequality)



Proof: (a) $\|c\vec{x}\| = \sqrt{\langle c\vec{x}, c\vec{x}\rangle} = \sqrt{\underbrace{c\,\bar{c}}_{=|c|^2}\langle \vec{x}, \vec{x}\rangle}$

$$= |c|\sqrt{\langle \vec{x}, \vec{x}\rangle} = |c|\,\|\vec{x}\|.$$

(b) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x}\rangle} \geq 0$ (by definition)

$\|\vec{x}\| = 0 \Leftrightarrow \langle \vec{x}, \vec{x}\rangle = 0$ iff $\vec{x} = \vec{0}$

(c) and (d) will be shown in the tutorial.

Orthogonality

Definition: Let V be an inner product space. We say $\vec{x}, \vec{y} \in V$ are orthogonal (or perpendicular) if $\langle \vec{x}, \vec{y} \rangle = 0$.

A subset $S \subset V$ is called orthogonal if any two distinct vectors in S are orthogonal.

A unit vector in V is a vector $\vec{x} \in V$ with $\|\vec{x}\| = 1$.

A subset $S \subset V$ is called orthonormal if S is orthogonal and all vectors in S are unit vectors.

e.g. Let H be the space of continuous complex-valued functions on $[0, 2\pi]$. We have inner product defined by:

$$\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt \quad \text{for } f, g \in H$$

For any $n \in \mathbb{Z}$ ← integer, let

$\sqrt{-1}$

$$f_n(t) = e^{int} \overset{\text{def}}{:=} \cos nt + i \sin nt \quad \text{for } t \in [0, 2\pi]$$

and consider $S = \{ f_n : n \in \mathbb{Z} \} \subset H$

For any $m \neq n$, we have:

$$\langle f_m, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\,\underset{\substack{\| \\ e^{-int}}}{\overline{e^{int}}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt = \frac{1}{2\pi}\left(\frac{1}{i(m-n)}\right)e^{i(m-n)t}\Big|_0^{2\pi}$$

$$= 0$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(m-n)t\,dt + i\frac{1}{2\pi}\int_0^{2\pi}\sin(m-n)t\,dt \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt$$

Also, $\langle f_n, f_n \rangle = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{int}\, \underset{\substack{\shortparallel \\ e^{-int}}}{\overline{e^{int}}}\, dt = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} 1\, dt = 1$

$\therefore$ S is orthonormal subset of H.

Definition: Let V be an inner product space. A subset of V is an **orthonormal basis** for V if it is an ordered basis which is orthonormal.

Proposition: Let V be an inner product space and $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be an orthogonal subset of V consisting of non-zero vectors.

Then: $\forall\, \vec{y} \in \mathrm{Span}(S)$,

$$\vec{y} = \sum_{i=1}^{k} \left(\frac{\langle \vec{y}, \vec{v}_i \rangle}{\| \vec{v}_i \|^2} \right) \vec{v}_i$$

Proof: Write $\vec{y} = \sum_{i=1}^{k} a_i \vec{v}_i$ for some $a_1, a_2, \ldots, a_k \in F$.

Take inner product with $\vec{v}_j$ on both sides gives:

$$\langle \vec{y}, \vec{v}_j \rangle = \sum_{i=1}^{k} a_i \langle \vec{v}_i, \vec{v}_j \rangle = a_j \|\vec{v}_j\|^2$$

Corollary 1: If, in addition to above, S is orthonormal, then $\forall \vec{y} \in \mathrm{Span}(S)$, $\vec{y} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$

<u>Corollary 2</u>: Let S be an orthogonal subset of an inner product space V consisting of non-zero vectors. Then, S is linearly independent.

<u>Proof</u>: If $\overset{\in \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_k\})}{\sum_{i=1}^{k} a_i \vec{v}_i} = \vec{0}$ for some $\vec{v}_1, \dots, \vec{v}_k \in S$ and $a_1, a_2, \dots, a_k \in F$,

By previous proposition,

$$a_i = \cancelto{0}{\langle \vec{0}, \vec{v}_i \rangle} \big/ \|\vec{v}_i\|^2 = 0 \quad \text{for } i = 1, 2, \dots, k. \qquad //$$

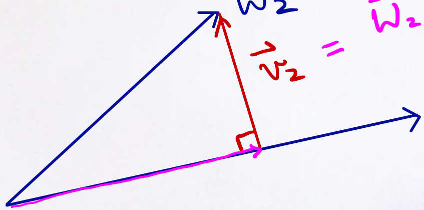
Prop: Let V be an inner product space and $S_n = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be a linearly independent subset of V. Define:

$$S_n' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \text{ where } \vec{v}_1 = \vec{w}_1 \text{ and}$$

for $k = 2, \ldots, n$,

$$\vec{v}_k \stackrel{\text{def}}{=} \vec{w}_k - \sum_{j=1}^{k-1} \left(\frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \right) \vec{v}_j$$

Then: S_n' is orthogonal and $\operatorname{Span}(S_n') = \operatorname{span}(S_n)$



$\vec{w}_2$

$\vec{v}_2 = \vec{w}_2 - c_1 \vec{v}_1$ where $c_1 = \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}$

$\vec{w}_1 = \vec{v}_1$